

Frictional Capital Reallocation II: Ex Post Heterogeneity*

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Abstract

In previous work we studied frictional markets for capital reallocation based on ex ante heterogeneity: similar firms enter the market with different capital stocks. Here we study ex post heterogeneity: firms with similar capital realize different productivity shocks. For different specifications, results are provided on existence, uniqueness, efficiency and the effects of monetary and fiscal policy. We show, e.g., how higher nominal rates can lower or raise investment, and can be desirable, despite hindering reallocation. The model can capture several stylized facts, e.g., misallocation appears countercyclical while capital reallocation and prices are procyclical. We also discuss how productivity dispersion may or may not accurately measure inefficiency and frictions.

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1 Introduction

This is our second of two papers on dynamic general equilibrium models of investment with frictional markets for existing capital (see Wright et al. 2017). While the specifications and the applications are different, as detailed below, naturally the papers overlap in motivation. In particular, any theory of capital should recognize that there are two components necessary for efficient economic performance: (i) getting the right amount of investment over time; and (ii) getting existing capital at a point in time into the hands of those best able to use it. Traditional macroeconomics concentrates on the former, but reducing capital mismatch via reallocation is currently receiving much attention. Of course, components (i) and (ii) are intimately related, because the ease with which used capital can be traded affects incentives for investment in new capital, just like the attributes of secondary markets for houses, cars and other assets influence demand and supply in primary markets.

The reallocation of existing capital is big.¹ Typically purchases of used capital are said to constitute 25% to 30% of total investment (Eisfeldt and Rampini 2006; Cao and Shi 2016; Cui 2017; Dong et al. 2016). In fact, given the data these studies do not include mergers, plus they ignore smaller firms and those that are not publicly traded. Also, the numbers are only for capital purchases, not rentals. Hence, the market for capital reallocation is important, and given that, we are interested in how it depends on fiscal and monetary policy. On the former, taxation has been shown to have big effects on capital accumulation, if not reallocation, e.g., by Cooley and Hansen (1992), Chari et al. (1994), McGrattan et al. (1997) and McGrattan (2012). On the latter, there is much precedent for

¹We emphasize reallocation across firms, but in principle one could also consider, e.g., moving capital within firms (Giroud and Mueller 2015), across sectors (Ramey and Shapiro 1998) or between countries (Caselli and Feyrer 2007).

studying inflation and capital accumulation, if not reallocation, e.g., including Tobin (1965), Sydrauski (1967), Stockman (1981) or Cooley and Hansen (1989), although they all use CIA or MUF models, while we take a New Monetarist approach that replaces such short cuts with more explicit frictions.²

Many people argue that real-world capital markets are neither perfectly competitive nor frictionless, including Kurman and Petrosky-Nadeau (2007), Gavazza (2010,2011*a,b*), Kurman (2014), Ottonello (2015), Cao and Shi (2016), Kurman and Rabinovitz (2018) and Horner (2018). Market imperfections potentially include information issues like adverse selection, financial constraints, the difficulty of finding an appropriate counterparty, and holdup problems coming from bargaining. We downplay the information issue (see Li and Whited 2014, Eisfeldt and Rampini 2008 and references therein) to concentrate on issues related to liquidity, search and bargaining. In our theory, the secondary market features bilateral trade, as in equilibrium search theory, and the use of assets to facilitate trade, as in modern monetary economics. Moreover, we consider random search and bargaining, as well as directed search and price posting.

As additional motivation for this approach, Ottonello (2015) compares models with and without search, argues the former fit the facts better, and shows these frictions are a relevant propagation mechanism for financial shocks (in his model these shocks account for 33% of fluctuations with search and 1% without). Horner (2018) shows that vacancy rates for industrial, office and retail real estate resemble unemployment rates, suggesting that search is as important for

²The New Monetarist literature is surveyed by Lagos et al. (2017) and Rocheteau and Nosal (2017). A difference is that we model firms trading inputs, while most of those papers focus on households trading goods, with a few exceptions. Work on capital based on Lagos and Wright (2005) includes Aruoba and Wright (2003), Lagos and Rocheteau (2008), Aruoba et al. (2011) and Andolfatto et al. (2016). Other work includes Shi (1998,1999*a,b*), Shi and Wang (2006), Menner (2006) and Berentsen et al. (2011), based on Shi (1997), and Molico and Zhang (2006), based on Molico (2006). Also related are models of the market for ideas, e.g., Silveira and Wright (2010), Chiu and Meh (2011) or Chiu et al. (2017). All of this is quite different from our approach, however, in terms of formal modeling and applications.

capital as it is for labor, and argues that rents on similar structures vary considerably, inconsistent with Walrasian pricing. In the market for aircraft, which has been featured in several studies, used sales are thrice new sales, and prices vary inversely with search time (Pulvino 1998; Gilligan 2004; Gavazza 2011*a,b*). Gavazza (2011a) in particular reports how market thickness affects trading frequency, average utilization, dispersion of utilization, prices, and dispersion of prices. He also emphasizes the importance of specificity: capital is customized to a particular firm, and it is nonrivial to find another with the same needs.

Any analysis of reallocation builds on gains from trade, with capital flowing from lower- to higher-productivity firms in the model, as the data (Maksimovic and Phillips 2001; Andrade et al. 2001; Schoar 2002). Our earlier paper studied ex ante heterogeneity, where firms had different capital stocks due to differences in investment abilities, so there were gains from trade between those with more and those with less, even if their productivities are the same. In fact, we imposed that some firms could not get any capital in the primary market, and thus had to obtain it all in the secondary market. Clearly, it is also relevant to study ex post heterogeneity. Here all firms can accumulate capital in primary markets, but face idiosyncratic productivity shocks generating gains from trade between those with low and high realizations. This seems rather more natural, is more in line with other research, and has several other advantages over the ex ante model.

In particular, ex post heterogeneity lets us dispense with a few awkward assumptions in the earlier model. There, since some firms had to acquire all their capital in the frictional secondary market, they would like to acquire enough to last a while. We precluded this by imposing that they could only store it for one period. Moreover, by assumption, they had to search for counterparties in the secondary market every period – i.e., they could not form long-term relationships, even though they would want to. This is less of an issue in the new formulation,

since firms do not have the same desire for long-term relationships when they can build their own capital in the primary market.³

While different, the models have a few common features, including decreasing returns, for the following reason: otherwise, given frictionless labor markets, for any two firms meeting in the secondary market, the efficient outcome is for the more productive one to get all the capital. With decreasing returns, the more productive one should get some but not necessarily all of it. Now, getting all of it may be interesting – it looks like a merger or acquisition – but that can also happen with decreasing returns, it just does not have to happen. Another common feature of the formulations is that they are built for tractability. The current model reduces to two simple equations determining investment and reallocation, which allow us to easily demonstrate how outcomes depend on policy.

As a preview, conditional on investment, reallocation is efficient at the Friedman rule, which is the limit $\iota \rightarrow 0$ where ι is the nominal interest rate. However, investment can be too high or low, depending on various factors, including the capital income tax τ . At $\tau = \iota = 0$, e.g., with random search and bargaining there is a Hosios (1990) condition for bargaining power, $\theta = \theta^*$, that delivers efficiency; this is different from the model with ex ante heterogeneity, where no θ delivers efficiency, as explained below. Also, increasing ι has two effects described heuristically by saying that money and capital are in one sense complements and in another sense substitutes, and the net effect dominates depends on details, as explained below. Also, interestingly, fiscal and monetary policy are not symmetric: ceteris paribus, with bargaining the optimal tax is $\tau^* > 0$ if $\theta > \theta^*$ and $\tau^* < 0$ if $\theta < \theta^*$, but $\iota^* > 0$ is optimal for both $\theta > \theta^*$ and $\theta < \theta^*$.

³There is actually still an incentive to have enduring relationships here, since these could in principle ameliorate search frictions, but we can let the probability of finding a counterparty be sufficiently big that this is not too important. The main point is that the previous model forced some firms to acquire all their capital in the secondary market, which is clearly an extreme assumption.

The theory provides insights into the measurement of misallocation. It might seem that dispersion in capital productivity across firms is a natural indicator of mismatch, but we show that various measures of productivity dispersion are, in general, imperfect indicators of the underlying frictions and welfare/output gaps. An increase in ι , e.g., which captures greater financial frictions, can reduce the dispersion measures.⁴ It also provides insights into observations deemed important in the literature. One is that reallocation is procyclical even though capital mismatch appears countercyclical (Eisfeldt and Rampini 2006; Cao and Shi 2016). Here, in good times, there may well be less incentive to reallocate capital due to less productivity dispersion, but there is also more capital, so reallocation can be greater. Similarly, the price of reallocated capital can be higher in goods times (Lanteri 2016), as can the ratio of spending on used capital to total investment (Cui 2016).

Further in terms of the literature, Ottonello (2015), Cao and Shi (2016), Dong et al. (2016), Kurman and Rabinovitz (2018) and Horner (2018) are all recent papers on capital using search, but with key differences – e.g., they do not study monetary models. Rocheteau et al. (2017) study a monetary model, but again with key differences – e.g., there firms buy capital from competitive suppliers rather than trading with each other in frictional markets. Also relevant is empirical work on how productivity differences depend on allocative efficiency, including Hsieh and Klenow (2009), Buera et al. (2011), Midrigan and Xu (2014), Cooper and Schott (2016), Ai et al. (2015) and David and Venkateswaran (2017). Our model is broadly consistent with their findings. In particular, Buera et al. (2011) find that financial frictions explain the empirical regularities by distorting the allocation of capital across firms, even if self-financing alleviates the

⁴Similar results obtain if we replace higher ι as a measure of financial frictions with lower debt limits. We also get similar results for other types of frictions, including lower arrival rates and a higher tax or bargaining wedge.

problem somewhat. Our emphasis on liquidity is qualitatively consistent with this finding.

To connect with research on OTC financial markets, consider Duffie et al. (2005), which is a search model where agents trade assets due to shocks to individual valuations. We have that, too, but with significant differences: The asset here is neoclassical capital, instead of “trees” bearing “fruit.” We have capital (and labor) used to produce numeraire goods as in standard growth theory, instead of agents consuming “fruit” directly. Our agents face a genuine liquidity problem because they must pay at least in part with retained earnings held at low yield, instead of transferable utility. Moreover, Duffie et al. (2005) restrict asset positions to $\{0, 1\}$, following Shi (1995) or Trejos and Wright (2005), while we eliminate any such restrictions, following a more modern approach discussed in the surveys cited in fn. 2. This makes our setup perhaps more comparable to the OTC model of Lagos and Rocheteau (2009), but in any case, the ideas here are quite similar to those in the literature on frictional asset markets.

To connect with other macro models, note that shutting down firms’ idiosyncratic shocks here eliminates the need for secondary markets and liquidity, thus reducing our formulation to a standard Real Business Cycle model – literally, the one in Hansen (1985), right down to the utility function. Therefore, instead of saying we extend standard monetary theory to investigate about capital reallocation, one can just as well say we extend more mainstream macro to include idiosyncratic shocks and frictional secondary markets with liquidity considerations. By comparison, Asker et al. (2014) present a relatively standard macro model where it takes one period to build capital, and productivity dispersion arises due to this lag. What is missing, relative to our environment, is a market where capital can be retraded after the shocks are realized, and where search, bargaining and liquidity all play interesting roles.

In what follows, Section 2 describes the environment. Sections 3 and 4 consider general notions of equilibrium with pure credit and with money. Sections 5 and 6 specialize the specification and analyze equilibrium with random search and bargaining, and with directed search and posting, respectively. Section 7 presents numerical results to discuss stylized facts and measurement issues. Section 8 concludes.

2 Environment

Time is discrete and continues forever. As shown in Figure 1, in each period t , two markets convene sequentially: a frictional decentralized market, or DM; and a frictionless centralized market, or CM. This alternating market structure, adapted from Lagos and Wright (2005), is ideal for our purposes because the CM and DM correspond nicely to primary and secondary capital markets. In the CM, agents consume a numeraire good x , supply labor hours h , and build capital k . Then in the DM, rather than having households trade consumption goods as in a typical monetary models, we have firms retrading capital (although sometimes for convenience we refer to the agents as the households that own firms rather than firms per se). All agents (firm owners) have utility $U(x, h) = u(x) - Ah$ and discount between one CM and the next at rate $\beta \in (0, 1)$; without loss of generality they do not discount between the DM and the CM.

At the beginning of the DM each firm realizes an idiosyncratic productivity shock $\varepsilon \geq 0$, with CDF $G(\varepsilon)$. While it may be interesting to allow persistence, mainly to facilitate analytic results these shocks are i.i.d. Thus, each firm has a technology $F(k, h, \varepsilon)$ for the subsequent CM, where $\forall(k, h)$ higher ε entails more output, and $F(\cdot)$ displays decreasing returns to scale in (k, h) for reasons discussed in the Introduction. This generates gains from reallocating capital in

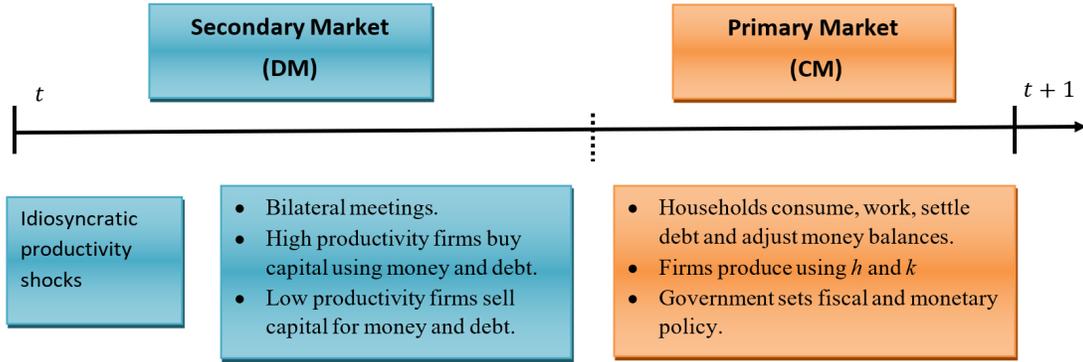


Figure 1: Time Line

the DM, which features random pair-wise meetings with α as the probability any agent meets a counterparty.⁵ Each meeting is characterized by a state $\mathbf{s} = (k_b, \varepsilon_b, k_s, \varepsilon_s)$, where k_b and ε_b are the capital and productivity of the buyer, while k_s and ε_s are the capital and productivity of the seller. Given the i.i.d. assumption, all agents acquire the same capital in the CM, so $k_b = k_s$ and the buyer is the one with the better shock, $\varepsilon_b > \varepsilon_s$.

In terms of efficiency, first, it is obvious that any firm in the CM with (k, ε) should set labor hours $h^*(k, \varepsilon)$ to satisfy

$$F_2[k, h^*(k, \varepsilon), \varepsilon]u'(x) = A, \quad (1)$$

with total h aggregating this across firms, and we assume the time constraint $h \in [0, 1]$ is slack. It is also obvious that any two firms meeting in the DM should set q to equate their marginal products, unless that is impossible because k_s is too low, whence the one with higher productivity should buy the other out, interpretable as a merger or acquisition.⁶ For simplicity, for now, suppose we

⁵As in standard macro/labor (e.g., Pissarides 2000) we do not take the concept of meetings literally as firms “bumping into each other” at random; it rather stands in for any friction making it hard to realize all gains from trade, something missing in standard GE theory.

⁶As Jovanovic and Rousseau (2002) say: “Used equipment and structures sometimes trade unbundled in that firm 1 buys a machine or building from firm 2, but firm 2 continues to

impose the Inada condition $F_1(k, h, \varepsilon) \rightarrow \infty$ as $k \rightarrow 0$, so that $q = q^*(\mathbf{s}) < k_s$ satisfies

$$F_1[k + q, h^*(k + q, \varepsilon_b), \varepsilon_b] = F_1[k - q, h^*(k - q, \varepsilon_s), \varepsilon_s]. \quad (2)$$

The *efficient reallocation condition* (2) plays an important role below.

With these properties of efficient outcomes in hand, consider a planner choosing a path for investment to maximize expected utility for the representative agent, subject to the search frictions, an initial k_0 , and resource feasibility given that government takes g_t units of numeraire each period. The problem is

$$\begin{aligned} W^*(k_0) &= \max_{k_{t+1}} \sum_{t=0}^{\infty} \beta^t [u(x_t) - Ah_t] & (3) \\ \text{st } x_t &= y_t + (1 - \delta)k_t - k_{t+1} - g_t \\ y_t &= (1 - \alpha) \int_0^{\infty} F[k_t, h^*(k_t, \hat{\varepsilon}), \hat{\varepsilon}] dG(\hat{\varepsilon}) \\ &\quad + \alpha \int_0^{\infty} \int_0^{\hat{\varepsilon}} F\{k_t + q^*(\hat{\mathbf{s}}), h^*[k_t + q^*(\hat{\mathbf{s}})], \hat{\varepsilon}\} dG(\tilde{\varepsilon}) dG(\hat{\varepsilon}) \\ &\quad + \alpha \int_0^{\infty} \int_{\hat{\varepsilon}}^{\infty} F\{k_t - q^*(\tilde{\mathbf{s}}), h^*[k_t - q^*(\tilde{\mathbf{s}})], \hat{\varepsilon}\} dG(\tilde{\varepsilon}) dG(\hat{\varepsilon}), \end{aligned}$$

where aggregate output y_t includes production by the $1 - \alpha$ measure of firms that did not have a DM meeting, the α measure that had a meeting and increased their capital, and the α measure that had a meeting and decreased their capital

Routine methods yield the investment Euler equation

$$\begin{aligned} r_t + \delta &= (1 - \alpha) \int_0^{\infty} F_1[k_{t+1}, h^*(k_{t+1}, \hat{\varepsilon}), \hat{\varepsilon}] dG(\hat{\varepsilon}) & (4) \\ &\quad + \alpha \int_0^{\infty} \int_0^{\hat{\varepsilon}} F_1\{k_{t+1} + q^*(\hat{\mathbf{s}}), h^*[k_{t+1} + q^*(\hat{\mathbf{s}})], \hat{\varepsilon}\} dG(\tilde{\varepsilon}) dG(\hat{\varepsilon}) \\ &\quad + \alpha \int_0^{\infty} \int_{\hat{\varepsilon}}^{\infty} F_1\{k_{t+1} - q^*(\tilde{\mathbf{s}}), h^*[k_{t+1} - q^*(\tilde{\mathbf{s}})], \hat{\varepsilon}\} dG(\tilde{\varepsilon}) dG(\hat{\varepsilon}). \end{aligned}$$

exist. At other times, firm 1 buys firm 2 and thereby gets to own all of firm 2's capital. In both markets, the traded capital gets a new owner." Typically the latter is interpreted as a merger or acquisition. For more on that, see Jovanovic and Rousseau (2002), Harford (2005) and references therein.

where r_t is given by $1 + r_t = u'(x) / \beta u'(x_{+1})$.⁷ On the LHS, $r + \delta$ is the marginal cost of investment, as agents postpone consumption and bear depreciation. The RHS is the marginal benefit, taking into account productivity shocks, plus the opportunity to reallocate capital after the shocks are realized. This *efficient reallocation condition* (4) plays an important role below.

Now that we have characterized hours, investment and reallocation, consumption x is simply given by the constraints in (3). We summarize as follows:⁸

Proposition 1 *The planner's outcome is described by a list of nonnegative and bounded time paths for $\langle x^*, h^*(\cdot), \hat{k}^*, q^*(\cdot) \rangle$ that for every date satisfy (1)-(4).*

3 Pure Credit Equilibrium

While our prime interest is in economies with credit frictions, it is useful a benchmark to consider those with perfect credit. In such economies, the firm with higher productivity in a DM meeting can purchase capital from the other one in exchange for debt d – i.e., a promise to pay the seller d units of numeraire in the next CM. Given quasi-linear utility, it is without loss of generality to restrict attention to one-period debt: agents would just as soon pay off d in the CM as roll it over, at least as long as $h \in [0, 1]$ is slack. Hence, the CM and DM value functions are denoted $W(a, k, \varepsilon)$ and $V(\hat{k}, \hat{\varepsilon})$, where (a, k, ε) includes an agent's financial asset position, real capital stock and productivity in the CM, while $(\hat{k}, \hat{\varepsilon})$ includes capital and productivity in the next DM. With perfect credit $a = -d$, but that changes when we introduce money.

⁷In equilibrium r_t is a real interest rate (more than one interest rate will be relevant below); here it is simply notation for the marginal rate of substitution. So, while (4) may look like a static conditions, it is not, since out of steady state k affects x and x affects r .

⁸By boundedness in the statement of Proposition 1, and other results below, we mean the standard transversality condition for k from neoclassical growth theory, and after adding money, an analogous condition for real balances (see, e.g., Rocheteau and Wright 2013). We do not dwell on this further, since this project mainly concerns steady states.

Consider first the CM – our primary market. The decision problem is

$$\begin{aligned}
W(a, k, \varepsilon) &= \max_{x, h, \hat{k}} \{u(x) - Ah + \beta \mathbb{E}_{\varepsilon} V_{+1}(\hat{k}, \hat{\varepsilon})\} & (5) \\
&\text{st } x = wh + a + \Pi(k, \varepsilon) + (1 - \delta)k - \hat{k} - T \\
\Pi(k, \varepsilon) &= \max_{\tilde{h}} \{F(k, \tilde{h}, \varepsilon) - w\tilde{h}\},
\end{aligned}$$

where in terms of numeraire the wage is w , the price of capital is 1, $\Pi(k, \varepsilon)$ is profit income, T is a lump-sum, and we omit t subscripts where the timing is obvious. From profit maximization, labor demand is

$$\tilde{h}(k, \varepsilon) = \arg \max_{\tilde{h}} \{F(k, \tilde{h}, \varepsilon) - w\tilde{h}\}. \quad (6)$$

This depends on w implicitly, even if not explicitly in the notation, but that is why we now use \tilde{h} , as opposed to h^* from the planner's problem.⁹

Using the constraints, we reduce (5) to

$$\begin{aligned}
W(a, k, \varepsilon) &= \frac{A}{w} [\Pi(k, \varepsilon) + a + (1 - \delta)k - T] + \max_x \left\{ u(x) - \frac{A}{w}x \right\} \\
&\quad + \max_{\hat{k}} \left\{ -\frac{A}{w}\hat{k} + \beta \mathbb{E}_{\varepsilon} V(\hat{k}, \hat{\varepsilon}) \right\}.
\end{aligned}$$

Given $h \in [0, 1]$ and nonnegativity constraints are slack, the FOC's are

$$x : \frac{A}{w} = u'(x) \quad (7)$$

$$\hat{k} : \frac{A}{w} = \beta \mathbb{E}_{\varepsilon} V_1(\hat{k}, \hat{\varepsilon}) \quad (8)$$

plus the budget equation, while the relevant envelope conditions are

$$W_1(a, k, \varepsilon) = \frac{A}{w} \quad (9)$$

$$W_2(a, k, \varepsilon) = \frac{A}{w} [F_1(k, \tilde{h}, \varepsilon) + 1 - \delta]. \quad (10)$$

⁹To mention a few details, we can allow agents to diversify their holdings across firms, which might seem desirable because of the ε shocks, but is actually not, again due to quasi-linear utility; hence we ignore it. Also, notice labor demand \tilde{h} by a firm does not generally coincide with the supply h of its owner – indeed, when labor is traded in the frictionless CM, theory does not pin down who works for whom. In future work it may be interesting to combine frictional capital and labor markets (Berentsen et al. 2010 and Dong and Xiao 2017 already integrate New Monetarist models with a Pissarides 2000 labor market, but not with capital).

These immediately imply some results that greatly enhance tractability, where part (i) follows directly from (8), and part (ii) from (9).¹⁰

Lemma 1 (i) \hat{k} is the same for all agents independent of (a, k, ε) ; (ii) W is linear in a .

Now consider the DM – our secondary market. Pairwise meetings are still characterized by $\mathbf{s} = (k_b, \varepsilon_b, k_s, \varepsilon_s)$, and in equilibrium the buyer is the one with $\varepsilon_b > \varepsilon_s$, because Lemma 2 implies $k_b = k_s$, although here we consider what happens off as well as on the equilibrium path. The trading surpluses are

$$\begin{aligned} S_b(\mathbf{s}) &= W[-d(\mathbf{s}), k_b + q(\mathbf{s}), \varepsilon_b] - W(k_b, \varepsilon_b) \\ S_s(\mathbf{s}) &= W[d(\mathbf{s}), k_s - q(\mathbf{s}), \varepsilon_s] - W(k_s, \varepsilon_s), \end{aligned}$$

where the buyer gets $q(\mathbf{s})$ in exchange for a promise $d(\mathbf{s})$. By the envelope conditions, these reduce these to

$$\begin{aligned} S_b(\mathbf{s}) &= \frac{A}{w} \{ \Pi[k_b + q(\mathbf{s}), \varepsilon_b] - \Pi(k_b, \varepsilon_b) + (1 - \delta) q(\mathbf{s}) - d(\mathbf{s}) \} \\ S_s(\mathbf{s}) &= \frac{A}{w} \{ \Pi[k_s - q(\mathbf{s}), \varepsilon_s] - \Pi(k_s, \varepsilon_s) - (1 - \delta) q(\mathbf{s}) + d(\mathbf{s}) \}. \end{aligned}$$

Intuitively, in S_b the buyer's profit and post-production capital are higher due to $q(\mathbf{s})$, but $d(\mathbf{s})$ has to be settled in the next CM; in S_s the situation is reversed. Also, again the constraint $q(\mathbf{s}) \leq k_s$ that may bind in some meetings, but standard conditions guarantee it is slack.

In the interest of manageability, assume both parties observe \mathbf{s} . (While it should go without saying that it is interesting to introduce private information, we think there is much to learn without that complication.) Then a variety

¹⁰If there were different types in the CM – e.g., firms were not ex ante homogeneous or shocks not i.i.d. – (\hat{z}, \hat{k}) may differ across types, but not within types, which is what matters for tractability. While this is due to quasi-linearity here, we can generalize $U(x, 1 - h)$ as in Wong (2016) to a larger class of functions and the results go through. Or, we can use any $U(x, 1 - h)$ if we assume indivisible labor and use employment lotteries, as mentioned in fn. 13).

of standard mechanisms can be used to determine the terms of trade, and we adopt Kalai's (1977) bargaining solution. This has several attractive properties, relative to generalized Nash, say, in models with liquidity considerations (Aruoba et al. 2007), so it is our benchmark, although with perfect credit Nash and Kalai give the same outcome. If θ is buyers' bargaining power, the solution is found by choosing (p, q) to maximize S_b subject to feasibility plus $S_b(\mathbf{s}) = \theta S(\mathbf{s})$, where $S(\mathbf{s}) = S_b(\mathbf{s}) + S_s(\mathbf{s})$. With perfect credit, $q = q^*(\mathbf{s})$ equates marginal products, using the same notation as (2) from the planner's problem. Then the Kalai condition $S_b(\mathbf{s}) = \theta S(\mathbf{s})$ determines

$$\begin{aligned} d^*(\mathbf{s}) &= (1 - \theta) \{ \Pi[k_b + q^*(\mathbf{s}), \varepsilon_b] - \Pi(k_b, \varepsilon_b) \} \\ &\quad + \theta \{ \Pi(k_s, \varepsilon_s) - \Pi[k_s - q^*(\mathbf{s}), \varepsilon_s] \} + (1 - \delta) q^*(\mathbf{s}). \end{aligned} \quad (11)$$

Hence debt is a weighted average of sellers' opportunity cost and buyers' gain in profit, plus the post-production value of the capital transfer, $(1 - \delta) q^*(\mathbf{s})$.

Every agent has a probability α of meeting a DM counterparty, drawn at random, for now, from the population. Therefore, after $\hat{\varepsilon}$ is realized, but before meetings occur, the expected payoff is

$$V(\hat{k}, \hat{\varepsilon}) = W(0, \hat{k}, \hat{\varepsilon}) + \alpha \int_0^{\hat{\varepsilon}} S_b(\hat{\mathbf{s}}) dG(\tilde{\varepsilon}) + \alpha \int_{\hat{\varepsilon}}^{\infty} S_s(\tilde{\mathbf{s}}) dG(\tilde{\varepsilon}). \quad (12)$$

The first term is the payoff from not trading in the DM, which is the continuation value in the CM. The second is the surplus from buying capital in the DM, where in equilibrium $\hat{\mathbf{s}} = (\hat{k}, \hat{\varepsilon}, \hat{k}, \tilde{\varepsilon})$ and $\tilde{\mathbf{s}} = (\hat{k}, \tilde{\varepsilon}, \hat{k}, \hat{\varepsilon})$ because the firm is a buyer when it realizes $\hat{\varepsilon}$ and meets a firm with $\tilde{\varepsilon} < \hat{\varepsilon}$. The third term is the surplus from selling in the DM because $\hat{\varepsilon} < \tilde{\varepsilon}$.

After reallocation, output is the same as y in the planner's problem (3), except \tilde{h} replaces h^* . Then goods market clearing requires $x + g + \hat{k} = y + (1 - \delta)k$, and labor market clearing can be ignored by Walras' Law (intuitively, wages adjust

to clear the market, where y depends on w because \tilde{h} does). We are now in position to define equilibrium. To conserve notation, let us not carry around labor demand in the definition, which is given by (6), but only the aggregate h . For initial conditions, let us start at $t = 0$ with all agents holding k_0 . Also, again one can replace Kalai bargaining with another mechanism and the definition would be the same except perhaps for the formula in (11).

Predicated on all this, we have the following:

Definition 1 *With perfect credit and bargaining, given the initial condition k_0 and time paths for policy $\langle g, T \rangle$, an equilibrium is a list of nonnegative and bounded time paths for $\langle x, h, \hat{k}, p(\cdot), q(\cdot), w \rangle$ such that for every date: (i) (x, h, \hat{k}) solves the CM maximization problem; (ii) $p(\cdot)$ and $q(\cdot)$ solve the DM bargaining problem; and (3) markets clear.*

Sometimes we concentrate on a simpler notion:

Definition 2 *With perfect credit and bargaining, given constant $\langle g, T \rangle$, a steady state is a time-invariant list $\langle x, h, \hat{k}, p(\cdot), q(\cdot), w \rangle$ that satisfies the equilibrium but not the initial conditions.*

As already mentioned, with perfect credit capital, reallocation is efficient given search frictions. To discuss investment we need the capital Euler equation. After routine algebra the result is

$$\begin{aligned}
r + \delta &= (1 - \alpha) \int_0^\infty \Pi_1(\hat{k}, \hat{\varepsilon}) dG(\hat{\varepsilon}) \\
&+ \alpha \int_0^\infty \int_0^{\hat{\varepsilon}} \left[\theta \Pi_1[\hat{k} + q(\hat{\mathbf{s}}), \tilde{\varepsilon}] + (1 - \theta) \Pi_1(\hat{k}, \hat{\varepsilon}) dG(\tilde{\varepsilon}) \right] dG(\hat{\varepsilon}) \\
&+ \alpha \int_0^\infty \int_{\hat{\varepsilon}}^\infty \left[(1 - \theta) \Pi_1[\hat{k} - q(\hat{\mathbf{s}}), \hat{\varepsilon}] + \theta \Pi_1(\hat{k}, \hat{\varepsilon}) \right] dG(\tilde{\varepsilon}) dG(\hat{\varepsilon}).
\end{aligned} \tag{13}$$

By virtue of the envelope theorem, we can replace $\Pi_1(\cdot)$ by $F_1(\cdot)$, making (13) look more like condition (4) for the planner. However, the conditions are still

different, due to θ , so equilibrium is not generally efficient, because of *holdup problems* in bargaining. In particular, comparing (13) to (4), the first line is the same, but the second is the same iff $\theta = 1$ and the third is the same iff $\theta = 0$.

We have more to say about this below, including comparison to existing results; for now, here is some intuition. On the one hand, $\theta < 1$ increases individual demand for k in the primary market, relative to the planner, because buying in the secondary market is less attractive when sellers extract part of the surplus by bargaining for higher d . On the other hand, $\theta > 0$ decreases demand for k in the primary market, because the option to sell it in the secondary market is less attractive when buyers extract part of the surplus by bargaining for lower d . The former channel leads to over-accumulation while the latter leads to under-accumulation of capital, suggesting that there is a $\theta^* \in (0, 1)$ that delivers efficiency, as per classic discussions in Mortensen (1982), Grout (1984) and Hosios' (1990). This is stated without proof here because it is a special case of other results derived below.

Proposition 2 *In pure credit equilibrium with bargaining, consumption, hours and capital reallocation are efficient conditional on investment, while investment is too high if $\theta < \theta^*$, too low if $\theta > \theta^*$, and just right if $\theta = \theta^*$.*

4 Monetary Equilibrium

It is well known (see the surveys cited in fn. 2) that money, or indeed any asset, can be valued for its liquidity only when credit is imperfect, and that this obtains when there is a lack of perfect commitment, so that than we cannot trivially assume agents honor debt obligations, plus a lack of perfect information, so that we cannot get them to honor said obligations by punishing those who renege. If sufficient information were available concerning those who renege, the threat

of taking away future credit could potentially be used to dissuade opportunistic default, as in papers following Kehoe and Levine (1983). We rule this out by assuming that information is insufficient to support any credit, precisely to get assets valued for their liquidity, although all we really need is that information is insufficient to emulate perfect credit (see Gu et al. 2016).

Alternatively, we could punish defaulters by taking away their future profits, as in work following Holmstrom and Tirole (1998). To make this imperfect, those papers typically assume only a fraction χ_1 of future profits are pledgeable – i.e., can be seized in the event of default. Additionally, agents can try to use current asset holdings to facilitate intertemporal exchange, as in work following Kiyotaki and Moore (1997). At the time of a DM transaction, our buyers hold k units of capital from the CM, and could in principle pledge a fraction χ_2 of that, plus a fraction χ_3 of newly-purchased capital q . We assume $\chi_1 = \chi_2 = 0$, although all we really need is to assume pledgeability is insufficient to emulate perfect credit (again see Gu et al. 2016), but in a nod to realism (see, e.g., Gavazza 2011a) we set $\chi_3 = 1$. Hence, buyers can secure some credit with newly-purchased capital at its post-production value, $(1 - \delta)w$, just like mortgages secure credit in housing markets.¹¹

Having $\chi_3 = 1$ is not enough to emulate perfect credit. Consider trying to buy q on credit using $(1 - \delta)q$ as collateral and nothing else. There is no deal, since the capital is worth $(1 - \delta)q$ to the seller even neglecting its use in production. Therefore buyers offering something by way of *down payment*. The only asset

¹¹Having new capital more pledgeable than old could be motivated by saying, e.g., that new capital is very easy to seize in the event of default, while old capital is very difficult, perhaps because we do not know where it is, or maybe what it is, or how good it is. Still, we think more research is warranted on similarities, differences and complementarities between various types of credit in our theories. Also, notice that securing credit with newly purchased q is equivalent to a rental agreement: a firm pays something up front to lease q units of capital, and returns $(1 - \delta)q$ after using it. In case this is not obvious, note that in a frictionless market returning $(1 - \delta)q$ is equivalent to keeping it and paying off a debt with the same value.

available to serve in this capacity here is fiat currency, by assumption, although yet again all we really need is that other assets that could serve in this capacity are at least somewhat scarce (Geromichalos et al. 2007; Lagos and Rocheteau 2008). So down payments must come out of cash brought into the DM from the previous CM, interpreted as *internal finance*. If m is an agent's nominal cash and ϕ its price in terms of numeraire x , then $z = \phi m$ denotes real balances. Now any DM meeting is characterized by $\mathbf{s} = (z_b, k_b, \varepsilon_b, z_s, k_s, \varepsilon_s)$, with a slight abuse of notation since now the z 's show up. Given \mathbf{s} , as before $q(\mathbf{s})$ is quantity while $d(\mathbf{s})$ is debt, and now $p(\mathbf{s})$ is a cash down payment.

The money supply follows $M_{+1} = (1 + \mu) M$, where μ is determined by policy, so the government's CM budget constraint becomes $g = T + \phi(M_{+1} - M)$.¹² The inflation rate is $\pi = \phi/\phi_{+1} - 1$. By the Fisher equation, $1 + \iota = (1 + r)(1 + \phi/\phi_{+1})$ gives the return on a nominal bond that is illiquid – i.e., cannot be traded in the DM – just like $1 + r = u'(x)/\beta u'(x_{+1})$ gives the return on a real bond that is illiquid. As usual, these bonds can be priced whether or not they trade in equilibrium: simply think of $1 + \iota_t$ as the amount of cash in the CM at $t + 1$ that makes you willing to give up a dollar in the CM at t , and $1 + r_t$ as the analog for numeraire. In steady state, $\pi = \mu$ and $1 + \iota = (1 + \mu)/\beta$, so it is equivalent to describe monetary policy by the choice of μ , π or ι . As is standard, we impose $\mu > \beta - 1$, which is equivalent in steady state to $\iota > 0$, but we also consider the limit $\mu \rightarrow \beta - 1$, or $\iota \rightarrow 0$, which is the Friedman rule.

The CM problem is similar to (5), except the budget equation becomes

$$x + (1 + \pi)\hat{z} = wh + a + \Pi(k, \varepsilon) + (1 - \delta)k - \hat{k} - T, \quad (14)$$

¹²In Section 5 we add profit taxes; we can do so here, too, but defer because the notation is already fairly intense. In any case, if $\mu > 0$ then newly-issued currency goes into general revenue, and it does matter here if this entails higher g or lower T (and symmetrically for $\mu < 0$). We mention this because some people dislike models where money is injected by lump sum transfers. We could also introduce government bonds and inject it by open market operations, as in Rocheteau et al. (2018), but that would be a distraction for present purposes.

and now $a = z - d$ is the net asset position, while $1 + \pi$ is today's relative price of real balances for the next DM. Applying similar methods to those used with pure credit, the key FOC's are

$$\hat{z} : \frac{A(1 + \pi)}{w} = \beta \mathbb{E}_\varepsilon V_1(\hat{z}, \hat{k}, \hat{\varepsilon}) \quad (15)$$

$$\hat{k} : \frac{A}{w} = \beta \mathbb{E}_\varepsilon V_2(\hat{z}, \hat{k}, \hat{\varepsilon}), \quad (16)$$

while the envelope conditions are the same as (9)-(10). Then the natural generalization of Lemma 1 is this:

Lemma 2 (i) (\hat{z}, \hat{k}) is the same for all agents, independent of (a, k, ε) ; (ii) W is linear in a .

Moving to the DM, the trading surpluses are now

$$\begin{aligned} S_b(\mathbf{s}) &= \frac{A}{w} \{ \Pi[k_b + q(\mathbf{s}), \varepsilon_b] - \Pi(k_b, \varepsilon_b) + (1 - \delta)q(\mathbf{s}) - p(\mathbf{s}) - d(\mathbf{s}) \} \\ S_s(\mathbf{s}) &= \frac{A}{w} \{ \Pi[k_s - q(\mathbf{s}), \varepsilon_s] - \Pi(k_s, \varepsilon_s) - (1 - \delta)q(\mathbf{s}) + p(\mathbf{s}) + d(\mathbf{s}) \}. \end{aligned}$$

(We continue to assume \mathbf{s} is common knowledge in meetings, but, as with perfect credit, it would be interesting to consider private information about k or ε , and now also z). There are three constraints: $q(\mathbf{s}) \leq k_s$, $d(\mathbf{s}) \leq (1 - \delta)q(\mathbf{s})$ and $p(\mathbf{s}) \leq z_b$. As before, standard conditions make the first one slack.

For the second constraint, note that neither buyers nor sellers care ex post (in the meeting) about the payment instrument as long as the constraints hold. Therefore a buyer may as well use all the credit he can before tapping his cash, but as remarked above, he still needs some cash as a down payment. We formalize this as follows:

Lemma 3 *With imperfect credit, in all DM trade $d(\mathbf{s}) = (1 - \delta)q(\mathbf{s})$ and $p(\mathbf{s}) > 0$.*

Based on these results, we can simplify the previous expressions to

$$S_b(\mathbf{s}) = \frac{A}{w} \{ \Pi[k_b + q(\hat{\mathbf{s}}), \varepsilon_b] - \Pi(k_b, \varepsilon_b) - \hat{p}(\mathbf{s}) \} \quad (17)$$

$$S_s(\mathbf{s}) = \frac{A}{w} \{ \Pi[k_s - \hat{q}(\mathbf{s}), \varepsilon_s] - \Pi(k_s, \varepsilon_s) + \hat{p}(\mathbf{s}) \}. \quad (18)$$

Notice $(1 - \delta)q(\mathbf{s})$ and $d(\mathbf{s})$ cancel in (17) and (18), although debt is still relevant, as it allows agents to economize on cash.

This brings us to the constraint, $p(\mathbf{s}) \leq z_b$. For at least some \mathbf{s} it must bind – i.e., the buyer must cash out – intuitively, because $\iota > 0$ makes currency a poor saving vehicle, so agents never hold more than they ever spend. For a wide class of solution concepts, including Kalai, Nash and simple strategic bargaining, as well as Walrasian pricing and more complicated mechanisms like the one in Hu et al. (2009), Gu and Wright (2016) and Zhu (2017) show this: (i) $p \leq z_b$ slack implies $q^*(\mathbf{s})$ is the same as perfect credit and the mechanism determines $p^*(\mathbf{s})$; (ii) $p \leq z_b$ binding implies $p(\mathbf{s}) = z_b$ and the mechanism determines $q(\mathbf{s}) < q^*(\mathbf{s})$. In general, there is a set \mathcal{B} such that $p \leq z_b$ binds iff $\mathbf{s} \in \mathcal{B}$, where $\iota > 0$ implies $\text{prob}(\mathbf{s} \in \mathcal{B}) > 0$ but not necessarily $\text{prob}(\mathbf{s} \in \mathcal{B}) = 1$.

With Kalai bargaining, in particular, $\mathbf{s} \notin \mathcal{B}$ implies the down payment is

$$p^*(\mathbf{s}) = (1 - \theta) \{ \Pi[k_b + q^*(\mathbf{s}), \varepsilon_b] - \Pi(k_b, \varepsilon_b) \} + \theta \{ \Pi(k_s, \varepsilon_s) - \Pi[k_s - q^*(\mathbf{s}), \varepsilon_s] \},$$

the same as (11); and $\mathbf{s} \in \mathcal{B}$ implies the quantity is determined by

$$z_b = (1 - \theta) \{ \Pi[k_b + q(\mathbf{s}), \varepsilon_b] - \Pi(k_b, \varepsilon_b) \} + \theta \{ \Pi(k_s, \varepsilon_s) - \Pi[k_s - q(\mathbf{s}), \varepsilon_s] \}.$$

While $V(\hat{z}, \hat{k}, \hat{\varepsilon})$ now depends on three arguments, it is still given by the RHS of (12), and CM output is the same as in the pure credit economy. Goods market clearing is also the same, money market clearing is simply $m = M$, and labor market clearing is again ignored by Walras' Law.

We are now in position to define monetary equilibrium, by which we mean an outcome with $z > 0$.¹³ For initial conditions all agents start with (z_0, k_0) , and monetary policy is specified as setting the growth rate of currency μ (one can also specify it as setting π or ι , but, as mentioned above, it does not matter in steady state). Then we have the following:

Definition 3 *With imperfect credit and bargaining, given (z_0, k_0) and time paths for $\langle \mu, g, T \rangle$, monetary equilibrium is a list of nonnegative and bounded paths for $\langle x, h, \hat{z}, \hat{k}, p(\cdot), q(\cdot), w \rangle$ such that for every date: (i) (x, h, \hat{z}, \hat{k}) solves the CM maximization problem; (ii) $p(\cdot)$ and $q(\cdot)$ solve the DM bargaining problem; and (3) markets clear.*

From what is known about monetary models, in general, there can be many dynamic equilibria based on beliefs, including, for some parameters, cyclic, chaotic and stochastic equilibria (e.g., Rocheteau and Wright 2013). This is an inescapable implication of taking liquidity seriously in dynamic general equilibrium, but while such outcomes are interesting, as are transitional dynamics, we sometimes concentrate on this simpler notion:

Definition 4 *With imperfect credit and bargaining, given constant $\langle \mu, g, T \rangle$, a monetary steady state is a time-invariant list $\langle x, h, \hat{z}, \hat{k}, p(\cdot), q(\cdot), w \rangle$ that satisfies the equilibrium but not the initial conditions.*

There are two Euler equations here, one for money and one for capital. To derive them, first, calculate the derivatives of $p(\cdot)$ and $q(\cdot)$. Supposing that $\mathbf{s} \in \mathcal{B}$

¹³As in any good model with fiat currency, there exists a nonmonetary equilibrium, which reduces here to a standard growth model, because the DM shuts down. Indeed, if we eliminate idiosyncratic and add aggregate productivity shocks, the nonmonetary equilibrium replicates exactly Hansen's (1985) real business cycle model. The only detail is that he does not start with $u(x) - Ah$; he derives it from general utility $U(x, h)$ by imposing indivisible labor, $h \in \{0, 1\}$, and incorporating employment lotteries à la Rogerson (1988). But the same method works here: assuming indivisible labor and lotteries, agents with any $U(c, h)$ act as if their utility functions were $u(x) - Ah$, leading to all same simplifications (see Rocheteau et al. 2008).

(the other case is similar), we have $\partial p/\partial z_b = 1$, $\partial p/\partial k_b = \partial p/\partial k_s = 0$, and

$$\frac{\partial q}{\partial z_b} = 1/D(\mathbf{s}) \quad (19)$$

$$\frac{\partial q}{\partial k_b} = (1 - \theta) \{ \Pi_1(k_b, \varepsilon_b) - \Pi_1[k_b + q(\mathbf{s}), \varepsilon_b] \} / D(\mathbf{s}) \quad (20)$$

$$\frac{\partial q}{\partial k_s} = \theta \{ \Pi_1[k_s - q(\mathbf{s}), \varepsilon_s] - \Pi_1(k_s, \varepsilon_s) \} / D(\mathbf{s}), \quad (21)$$

where

$$D(\mathbf{s}) \equiv (1 - \theta) \Pi_1[k_b + q(\mathbf{s}), \varepsilon_b] + \theta \Pi_1[k_s - q(\mathbf{s}), \varepsilon_s]. \quad (22)$$

Then evaluate these at $\hat{\mathbf{s}} = (\hat{z}, \hat{k}, \hat{\varepsilon}, \hat{z}, \hat{k}, \hat{\varepsilon})$ and insert them into the derivatives of $V(\cdot)$ wrt the CM choice variables (\hat{z}, \hat{k}) . Finally, insert those into the FOC's (15)-(16) to get the Euler equations.

For money, the result is

$$\frac{1 + \pi}{w} = \frac{\beta}{w_{+1}} \left[1 + \alpha \int_0^\infty \int_0^{\hat{\varepsilon}} \Lambda(\hat{\mathbf{s}}) dG(\tilde{\varepsilon}) dG(\hat{\varepsilon}) \right], \quad (23)$$

where

$$\Lambda(\hat{\mathbf{s}}) \equiv \frac{\theta \{ \Pi_1[\hat{k} + q(\hat{\mathbf{s}}), \hat{\varepsilon}] - \Pi_1[\hat{k} - q(\hat{\mathbf{s}}), \tilde{\varepsilon}] \}}{D(\hat{\mathbf{s}})}. \quad (24)$$

Observe that $w_{+1}/\beta w = u'(x)/\beta u'(x_{+1}) = 1 + r$ and $(1 + \pi)(1 + r) = 1 + \iota$. Therefore, (23) can be written

$$\iota = \alpha \int_0^\infty \int_0^{\hat{\varepsilon}} \Lambda(\hat{\mathbf{s}}) dG(\tilde{\varepsilon}) dG(\hat{\varepsilon}). \quad (25)$$

The LHS of (25) is the nominal interest rate, which is the marginal cost of holding cash; the RHS is the marginal benefit, which is the expectation the firm draws $\hat{\varepsilon}$ and meets a counterparty with $\tilde{\varepsilon} < \hat{\varepsilon}$, times the former's share θ of the increase in total surplus due to his liquidity. In fact, $\Lambda(\hat{\mathbf{s}}) = \partial S_b/\partial z_b$ is the Lagrange multiplier on $p \leq z_b$, usually called the *liquidity premium* in related models of consumers buying goods (again see the surveys cited in fn. 2).

From (25), conditional on \hat{k} reallocation is efficient iff $\iota = 0$, since having no cost to liquidity is like having perfect credit. But, as in the pure credit economy,

reallocation is efficient only conditional on investment. Also, in general, $\iota > 0$ is a *wedge*, and the distortion due to $\iota > 0$ is bigger when θ is smaller. This is because smaller θ implies that buyers get a smaller share of the surplus, and hence recoup less of their cost of investing in liquidity – a holdup problem in money demand.

Similarly, the capital Euler equation is

$$\begin{aligned}
r + \delta &= (1 - \alpha) \int_0^\infty \Pi_1(\hat{k}, \hat{\varepsilon}) dG(\hat{\varepsilon}) \\
&+ \alpha \int_0^\infty \int_0^{\hat{\varepsilon}} \Pi_1[\hat{k} + q(\hat{\mathbf{s}}), \tilde{\varepsilon}] \Omega(\hat{\mathbf{s}}) dG(\tilde{\varepsilon}) dG(\hat{\varepsilon}) \\
&+ \alpha \int_0^\infty \int_{\hat{\varepsilon}}^\infty \Pi_1[\hat{k} - q(\tilde{\mathbf{s}}), \hat{\varepsilon}] \Gamma(\tilde{\mathbf{s}}) dG(\tilde{\varepsilon}) dG(\hat{\varepsilon}),
\end{aligned} \tag{26}$$

where $\hat{\mathbf{s}} = (\hat{z}, \hat{k}, \hat{\varepsilon}, \hat{z}, \hat{k}, \tilde{\varepsilon})$, $\tilde{\mathbf{s}} = (\hat{z}, \hat{k}, \tilde{\varepsilon}, \hat{z}, \hat{k}, \hat{\varepsilon})$, and

$$\Omega(\hat{\mathbf{s}}) \equiv \frac{(1 - \theta) \Pi_1(\hat{k}, \hat{\varepsilon}) + \theta \Pi_1[\hat{k} - q(\hat{\mathbf{s}}), \tilde{\varepsilon}]}{D(\hat{\mathbf{s}})} \tag{27}$$

$$\Gamma(\tilde{\mathbf{s}}) \equiv \frac{(1 - \theta) \Pi_1[\hat{k} + q(\tilde{\mathbf{s}}), \hat{\varepsilon}] + \theta \Pi_1(\hat{k}, \tilde{\varepsilon})}{D(\tilde{\mathbf{s}})} \tag{28}$$

The terms $\Omega(\hat{\mathbf{s}})$ and $\Gamma(\tilde{\mathbf{s}})$ capture capital holdup problems in monetary equilibrium, generalizing Section 3. If $\theta < 1$ then $\Omega(\hat{\mathbf{s}}) > 1$, increasing CM demand for k because buying it in the DM is less attractive when sellers extract more surplus. If $\theta > 0$ then $\Gamma(\tilde{\mathbf{s}}) < 1$, decreasing CM demand for k because the option to sell in the DM is less attractive when buyers extract more surplus.

From (13) and (26), k is efficient at $\Omega(\hat{\mathbf{s}}) = \Gamma(\tilde{\mathbf{s}}) = 1$, and $\theta > 0$ implies under-investment while $\theta < 1$ implies over investment. However, what is perhaps most interesting is the interaction between the wedges in the demand for capital and money. It remains true that there exists $\theta^* \in (0, 1)$ such that, as long as the liquidity wedge is shut down by setting $\iota = 0$, equilibrium attains the first best at $\theta = \theta^*$. Also, although we cannot get the first best at $\iota > 0$, there is a θ^{**} such equilibrium attains the second best at $\theta = \theta^{**}$. Furthermore, monetary policy affects investment through interesting and somewhat novel channels.

First, higher ι increases the demand for k in the CM, because buying it in the DM is less attractive when z is more expensive. This is reminiscent of the Mundell-Tobin effect of inflation on the demand for capital and other nonmonetary assets, although our microfoundations are rather different. Second, higher ι decreases demand for z , and that reduces investment in the CM, because the option value of selling k in the DM is less attractive when liquidity is more tight (i.e., in real terms, there is less cash in the market). This is reminiscent of Keynesian suggestions that lowering nominal interest rates will stimulate real investment, although again the microfoundations are very different. Which effect dominates? That depends on details analyzed below.¹⁴

In terms of the literature, Kurman and Rabinovitz (2018) (and earlier references cited therein) have k holdup problems but not m holdup problems, because they do not consider monetary models. The monetary theory in the surveys cited in fn. 2 have m holdup problems but not k holdup problems, with a few exceptions, like Aruoba et al. (2011), but there agents trade consumption in the DM, not capital, so the investment channels discussed here are absent. Our previous model (Wright et al. 2017) has m and k holdup problems, but the implications are very different, because there some agents bring k but not m to the DM while others bring m but not k , in contrast to the current environment, where the representative agent brings both. Hence, our earlier setup is more like labor models by Masters (1998, 2011) or Acemoglu and Shimer (1999), where firms invest in physical capital and workers in human capital.

This is not a minor technicality, but makes major differences. For one, when some agents bring k and others bring m , there is no $\theta = \theta^*$ that attains the first

¹⁴Heuristically, the two effects can be described by saying that money and capital are in one sense complements and in another sense substitutes. In similar models with consumer buying goods, e.g., in Lester et al. (2013) money and other assets as substitutes, in Li and Li (2013) they are complements, and in He et al. (2013) it depends on parameters.

best at $\iota = 0$. For another, as in the earlier paper, the k holdup problem here tends to reduce investment, while the m holdup problem tends to reduce real balances and hence reallocation, but importantly, the m holdup problem here also tends to increase investment in primary market. This last effect, which we consider a big part of the contribution arising from the theory, simply cannot emerge when agents bring either k or m but not both.¹⁵

Proposition 3 *In monetary equilibrium with bargaining and imperfect credit, consumption, if $\iota = 0$, hours and capital reallocation are efficient conditional on investment, while investment is too high if $\theta < \theta^*$, too low if $\theta > \theta^*$, and just right if $\theta = \theta^*$.*

5 A Convenient Parameterization

General versions of the models presented above can be studied quantitatively, in steady state or otherwise, with or without i.i.d. shocks, and with or without ex ante heterogeneity. That is deferred to future work; instead we concentrate here on simpler specifications that yield more analytic results, because it seems a good way of developing and communicating salient economic ideas. Thus, we now assume $\varepsilon \in \{\varepsilon_H, \varepsilon_L\}$, with $\varepsilon_H > \varepsilon_L$, where $\text{prob}(\varepsilon_L) = \gamma_L$ and $\text{prob}(\varepsilon_H) = \gamma_H = 1 - \gamma_L$. This is convenient because it implies $p \leq z_b$ binds in every DM trade, clearly, because it must bind in some meetings and now there is only one relevant type of meeting – one where a firm with ε_H meets a firm with ε_L . Moreover, since there is only one type of meeting with trade, the bargaining solution for

¹⁵In terms of a somewhat different literature, this novel result is somewhat related to work on household production, e.g., Burdett et al. (2016). In such models, inflation is a tax on monetary exchange, and hence on market relative to home economic activity. Hence, higher ι gives households more incentive to do their own cooking, cleaning, child care, etc., rather than trying to acquire similar goods and services on the market. Similarly, raising ι here gives firms more incentive to accumulate their own capital in the primary market, rather than trying to acquire similar capital on the secondary market.

q is a number rather than a function $q(\mathbf{s})$, and that allows us to easily depict equilibrium in (k, q) space. It also means $\alpha_H = \alpha\gamma_L$ and $\alpha_L = \alpha\gamma_H$, although we could generalize this as in, e.g., Pissarides (2000), by letting $\alpha_H = \xi(\gamma_H, \gamma_L)/\gamma_H$ and $\alpha_L = \xi(\gamma_H, \gamma_L)/\gamma_L$ for any function $\xi(\cdot)$ with constant returns. This would be especially interesting in an extension incorporating entry (costly participation in the DM) for buyers or for sellers, but that is left as an exercise. Furthermore, we now specialize the general formulation by considering a quasi-linear technology, $F(k, h, \varepsilon) = \varepsilon f(k) + h$. This is convenient because it means $w = 1$, and that simplifies several calculations. We also introduce tax τ on capital income to say more about interactions between monetary and fiscal policy. Obviously we could also add a tax on labor income, but that is also left as an exercise.

To begin, consider a planner problem like (3), except output is

$$y = Cf(k) + \xi\varepsilon_H f(k+q) + \xi\varepsilon_L f(k-q) + h,$$

where $\xi = \gamma_H\alpha_H = \gamma_L\alpha_L$ is the number of firms that trade in the DM, while $C = \gamma_L(1-\alpha_L)\varepsilon_L + \gamma_H(1-\alpha_H)\varepsilon_H$ captures productivity for the rest. The FOC's for q and k are

$$0 = \varepsilon_H f'(k+q) - \varepsilon_L f'(k-q) \tag{29}$$

$$r + \delta = Cf'(k) + \xi\varepsilon_H f'(k+q) + \xi\varepsilon_L f'(k-q), \tag{30}$$

which are obviously simpler than the general case. At this point we also impose steady state, so that $r = 1/\beta - 1$ is constant.¹⁶

¹⁶Different from the general model, (30) is basically a static condition: while r is still given by $1+r = u'(x)/\beta u'(x_{+1})$, with $w = 1$ we get $u'(x) = A$ in and out of steady state. This means the economy can jump to steady state in one period *unless* $h \in [0, 1]$ binds. To see what this means, consider a model with no DM – i.e., the standard growth model with utility and production functions both linear in h . Under mild conditions there is a unique steady state \bar{k} (note that 0 is not generally a steady state because output can be produced with h even at $k = 0$). At \bar{k} , x solves $u'(\bar{x}) = A$, and h solves $\bar{h} = \bar{x} + \delta\bar{k} - f(\bar{k})$ as long as $\bar{h} \leq 1$, which we assume. Suppose the initial k_0 is below \bar{k} , but close. Then we jump to \bar{k} in one period by setting

Condition (29) defines $q = Q(k)$ and (30) defines $k = K(q)$, two (single-valued) functions. We call $k = K(q)$ the IS curve, a standard name for the investment Euler equation in the literature, and call $q = Q(k)$ the CR curve, for capital reallocation, for obvious reasons. Efficiency obtains in (k, q) space when they cross. Their slopes are given by

$$\begin{aligned} \frac{\partial q}{\partial K|_{CR}} &= \frac{\Phi(k, q)}{\varepsilon_H f''(k+q) + \varepsilon_L f''(k-q)} \\ \frac{\partial q}{\partial K|_{IS}} &= \frac{\varepsilon_H f''(k+q) + \varepsilon_L f''(k-q) + C f''(k) / \xi}{\Phi(k, q)} \end{aligned}$$

where we define

$$\begin{aligned} \Phi(k, q) &\equiv \varepsilon_L f''(k-q) - \varepsilon_H f''(k+q) \\ &= f'(k+q) f''(k-q) - f'(k-q) f''(k+q), \end{aligned}$$

with the second line following from (29). Hence, when they cross, IS and CR both slope down if $\Phi(k, q) > 0$ and up if $\Phi(k, q) < 0$.

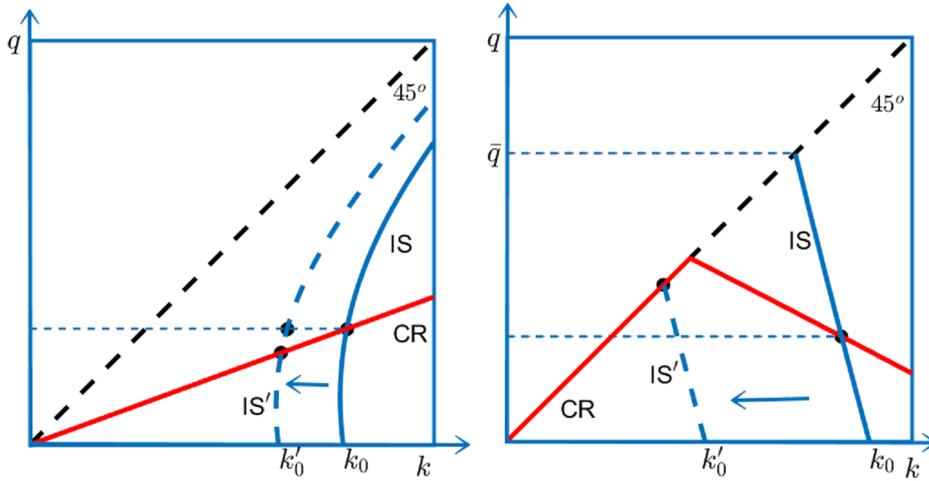


Figure 2: IS and CR: $\Phi < 0$ (left) and $\Phi > 0$ (right)

$h_0 = \bar{x} + \bar{k} - f(k_0) - (1 - \delta)k_0$. Now suppose k_0 is so low that $\bar{x} + \bar{k} - f(k_0) - (1 - \delta)k_0 > 1$, and the hours constraint initially binds. Then the transition has $h_t = 1$, with x_t and k_t determined in the obvious way, for $t = 1, 2, \dots$ until we reach k such that $h = \bar{x} + \bar{k} - f(k) - (1 - \delta)k \leq 1$, whence we jump to \bar{k} .

Both cases are possible, as shown for two numerical examples in Figure 2.¹⁷ Of course the curves lie below the 45° line, and as an aside, notice that in the right panel the CR curve yields $q = k$ (a corner solution for reallocation) when k is low. Although this is not case at the intersection of IS and LM in the example shown in the left panel, we can raise $r + \delta$ to shift IS left, and get $q = k$ at the new intersection, which as mentioned above can be interpreted as a merger or acquisition. In any event, from here on, we generally impose $r + \delta < (C + \xi\varepsilon_H + \xi\varepsilon_L) f'(0)$, since otherwise $k = 0$, and production ensues using only h , which is not a big problem, but evidently makes things less interesting. Then we have the following result:

Proposition 4 *The IS and RB curve implied by the planner problem cross uniquely in (k, q) space at $k > 0$ and $q > 0$.*

Proof: Let $\Delta = (\partial q / \partial K)_{IS} - (\partial q / \partial K)_{CR}$ and notice

$$\Delta = \frac{\varepsilon_H f''(k+q) + \varepsilon_L f''(k-q) + C f''(k) / \xi}{\Phi(k, q)} - \frac{\Phi(k, q)}{\varepsilon_H f''(k+q) + \varepsilon_L f''(k-q)}$$

Suppose $\Phi < 0$. Letting \approx indicate both sides take the same sign, we have

$$\begin{aligned} \Delta &\approx [\varepsilon_H f''(k+q) + \varepsilon_L f''(k-q)]^2 \\ &\quad + [\varepsilon_H f''(k+q) + \varepsilon_L f''(k-q)] C f''(k) / \xi - \Phi(k, q)^2 \\ &> [\varepsilon_H f''(k+q) + \varepsilon_L f''(k-q)]^2 - \xi \Phi(k, q)^2. \end{aligned}$$

Since the RHS is easily shown to be positive, when the both curve slope up IS is steeper than CR when they cross.

¹⁷The left panel uses $f(k) = k^{1/3}$, plus $r = 0.01$, $\delta = 0.02$, $\gamma_H = \gamma_L = 0.5$, $\varepsilon_L = 0.6$, $\varepsilon_H = 1$ and $\tau = 0.3$ (solid line) or 0.4 (dashed line); the right panel uses $f(k) = k - 0.4k^2 + 0.01k^3$, which is increasing and concave over the relevant range, if not globally, plus $r = \delta = 0.1$, $\gamma_H = \gamma_L = 0.5$, $\varepsilon_L = 0.3$, $\varepsilon_H = 1$ and $\tau = 0$ (solid line) or 0.5 (dashed line).

Now suppose $\Phi > 0$. Then we have

$$\begin{aligned}\Delta &\approx -[\varepsilon_H f''(k+q) + \varepsilon_L f''(k-q)]^2 \\ &\quad -[\varepsilon_H f''(k+q) + \varepsilon_L f''(k-q)]Cf''(k)/\xi + \Phi(k,q)^2 \\ &< -[\varepsilon_H f''(k+q) + \varepsilon_L f''(k-q)][\varepsilon_H f''(k+q) + \varepsilon_L f''(k-q)] + \Phi(k,q)^2.\end{aligned}$$

Since the RHS is now easily shown to be negative, when the both slope down IS is again steeper than CR when they cross. Hence, in either case they cannot cross more than once.

To show they must cross, first notice that CR satisfies $Q(0) = 0$ and $0 < Q(k) \leq k \forall k > 0$, and that as $k \rightarrow \infty$, $k - Q(k) \rightarrow \infty$ because $\varepsilon_H f'[k + Q(k)] > \varepsilon_H f'(k) \rightarrow 0$, implying $\varepsilon_L f'[k - Q(k)] \rightarrow 0$. Similarly, the IS curve satisfies $K(0) > 0$ and $K(q) - q \rightarrow c < \infty$ as $q \rightarrow \infty$. Now having the curves cross is equivalent to the existence of a solution to

$$Q \circ K(q) - q = 0, \tag{31}$$

where \circ denotes the composite of two functions. Notice $Q \circ K(0) > 0$. Moreover, as $q \rightarrow \infty$ we have

$$Q \circ K(q) - q = Q \circ K(q) - K(q) + K(q) - q \rightarrow c - \infty = -\infty.$$

Hence, there exists $\tilde{q} > 0$ solving (31), which means the curves cross at $k = K(\tilde{q})$ and $q = \tilde{q}$. ■

While it is not needed in general, sometimes in what follows we impose

$$\Phi(k,q) < 0, \tag{Condition F}$$

so that IS and CR slope up when they cross. This seems like the more natural case, and always holds for, e.g., $f(k) = k^\eta$ with $\eta \in (0,1)$, suggesting that Condition F is not especially stringent. As mentioned above, increasing $r + \delta$

shifts IS left, so k decreases. In addition, q decreases if IS and CR slope up, and q increases if they slope down (see Figure 2). While there is no big problem with q and k moving in opposite directions, Condition F tells the conditions under which they move in the same direction.

Let us now move to monetary equilibrium (with perfect credit left as an exercise). As mentioned, here we introduce a tax τ on capital income, so the CM constraint becomes

$$x + (1 + \pi)\hat{z} = h + a + (1 - \tau)\varepsilon f(k) + (1 - \delta)k - \hat{k} - T.$$

As also mentioned, there is only one relevant kind of meeting in the DM, and $p \leq z_b$ always binds, so the bargaining solution becomes

$$\frac{z}{1 - \tau} = (1 - \theta)\varepsilon_H[f(k + q) - f(k)] + \theta\varepsilon_L[f(k) - f(k - q)], \quad (32)$$

where we point out the appearance of τ on the LHS. Finally, the Euler equations become

$$\iota = \xi\Lambda(k, q) \quad (33)$$

$$\frac{r + \delta}{1 - \tau} = C f'(k) + \xi\varepsilon_H f'(k + q)\Omega(k, q) + \xi\varepsilon_L f'(k - q)\Gamma(k, q) \quad (34)$$

where $\Lambda(k, q)$, $\Omega(k, q)$ and $\Gamma(k, q)$ are simplified versions of the wedges defined in Section 4, and here we write the arguments as (k, q) rather than \mathbf{s} , again because there is only one relevant kind of DM meeting.

Like the planner's problem, equilibrium is recursive: first (33) and (34) define the CR and IS curves, intersections of which yield (k, q) ; then we solve for z , x , h etc. However, in a monetary economy our CR is nothing more nor less than the LM curve from traditional Keynesian analysis – hence, we may as well call it that, even though we still plot the curves in (k, q) space, different from traditional Keynesian graphical analyses that focus on output vs interest rates. Of course,

the reason we focus on this space is that k and q capture our two main objects of interest, capital accumulation and capital reallocation.

Moreover, while our methods for characterizing monetary equilibrium are similar to those used for the planner's problem, the results are slightly different. First, monetary equilibrium exists only for ι below a threshold $\bar{\iota}$, because if cash is taxed too heavily the secondary shuts down, and agents must get all their capital in the primary market. Second, uniqueness is not obvious because there are complementarities at work – if there is more cash in the market, in real terms, you may want to bring more capital, and vice versa – although we can establish uniqueness under some conditions. The proof of the following is in the Appendix.

Proposition 5 *With imperfect credit and bargaining, given $F(k, h, \varepsilon) = \varepsilon f(k) + h$ and $\varepsilon \in \{\varepsilon_H, \varepsilon_L\}$, a monetary steady state exists iff*

$$\iota < \bar{\iota} \equiv \frac{\alpha_H \gamma_H \theta (\varepsilon_H - \varepsilon_L)}{(1 - \theta) \varepsilon_H + \theta \varepsilon_L}.$$

It is unique if either θ is not too small or ι is not too big.

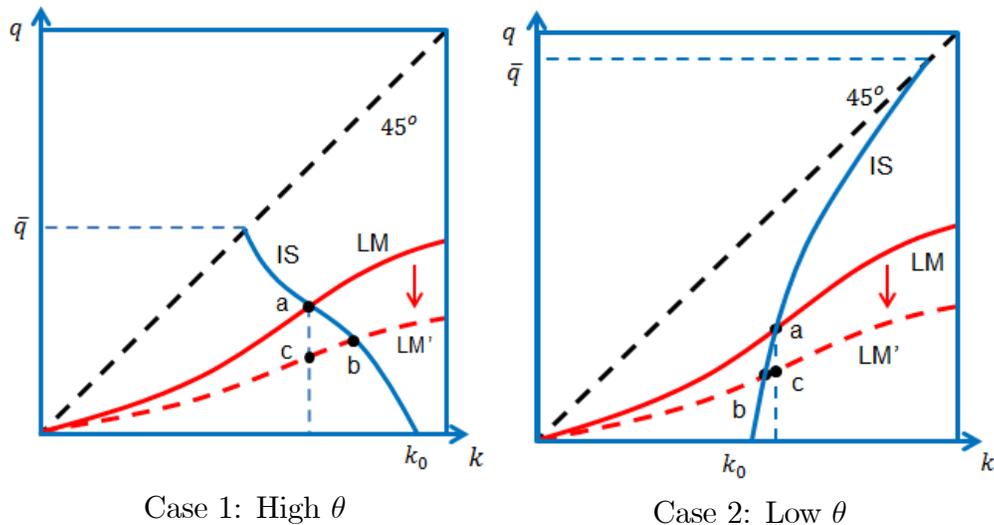


Figure 3: Monetary Policy, Increasing ι

The results are illustrated in Figures 3 and 4. Similar to CR in the planner's problem, the LM curve starts at $(0,0)$, lies below the 45° line, and while it is not monotone increasing in general, it is under Condition F. Different from the planner's problem, even under Condition F the IS curve can be increasing, decreasing, or nonmonotone. This is related to the discussion in Section 4: first, there is less need to bring k from the CM it is easier to get q in the DM, which tends to make IS decreasing, as in the left panels of Figures 2 and 3; second, higher q means a higher value to selling capital in the DM, which tends to make IS increasing, as in the right panels. On net, if θ is big, firms selling k in the DM get little surplus, so the first effect dominates and IS slopes down, and if θ is small, firms buying k in the DM get very surplus, so the second effect dominates and IS slopes up. Obviously this important impact of θ is absent in the planner's problem.

Figure 3 shows the effects of monetary policy. An increase in ι does not

affect IS, but rotates CR clockwise until ι hits $\bar{\iota}$, at which point it hits the horizontal axis and monetary equilibrium breaks down. For $\iota < \bar{\iota}$, as it increases, steady state moves from point a to b in Figure 2. Hence, q decreases, while k decreases or increases as the IS slopes up or down, Hence, at least for for some parameters, lower nominal interest (or inflation or money growth) rates increase capital investment. This is consistent with conventional Keynesian doctrine but the logic is different: here lower ι reduces the cost of liquidity, which facilitates trade in the secondary market, and that raises the option value of investing in the primary market. To see the multipliers at work, observe that an increase in ι would move us from a to c if k were fixed, but since k in fact reacts, we move to b , which attenuates the fall in q in the left panel and accentuates it in the right panel.

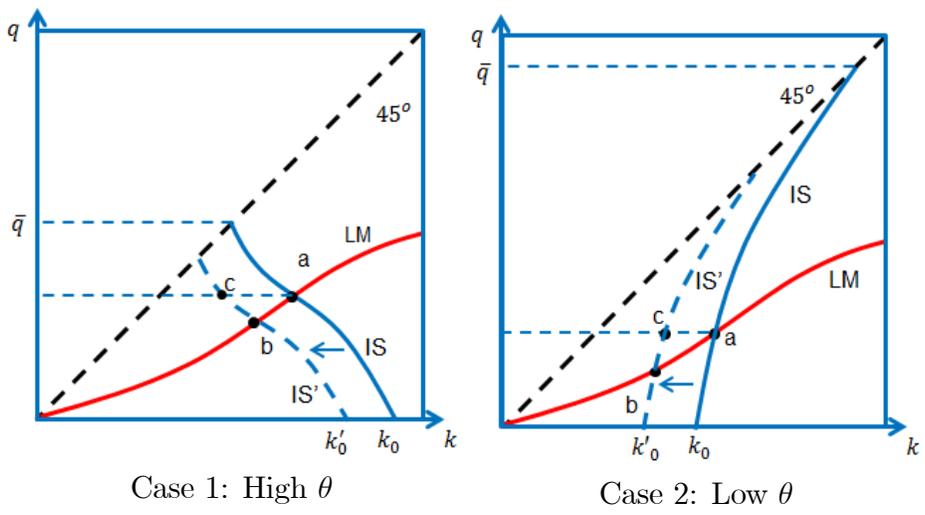


Figure 4: Fiscal Policy, Increasing τ

Figure 4 similarly shows the effects of fiscal policy, i.e., an increase in the profit tax τ . This shifts the IS curve to the left but does not affect CR, so q and k both decrease, regardless of whether IS slopes up or down, given LM slopes up.

To see the fiscal multipliers at work, in this case, observe that an increase in τ would move us from point a to c if q were fixed, but since q in fact reacts, we instead move to b , which attenuates the fall in k in the left panel and accentuates it in the right panel of Figure 3. Hence, fiscal as well as monetary policy can be analyzed conveniently and clearly using simple graphs, similar to those used in common undergraduate macro courses, although obviously with different (and we think better) microfoundations.¹⁸

As in the general model, if we set $\iota = \tau = 0$ then k is efficient in monetary equilibrium here iff $\theta = \theta^*$, where

$$\theta^* = \frac{\varepsilon_L [f'(k - q) - f'(k)]}{(\varepsilon_H - \varepsilon_L) f'(k)} \quad (35)$$

and (k, q) on the RHS are the solutions to the planner's problem. If (35) is violated, efficiency obtains at $\iota = 0$ and $\tau = \tau^*$ where

$$\tau^* = 1 - \frac{r + \delta}{\Delta}, \quad (36)$$

where

$$\begin{aligned} \Delta = & \varepsilon_H f'(k + q) + (1 - \theta \alpha_H) \gamma_H \varepsilon_H f'(k) \\ & + [1 - (1 - \theta) \alpha_L] \gamma_L \varepsilon_L f'(k). \end{aligned}$$

Notice that Δ decreases with θ , which implies that τ^* decreases with θ . In particular, if $\theta > \theta^*$ we need a subsidy $\tau^* < 0$; otherwise we need a tax $\tau^* > 0$.

These observations prove the following result:

¹⁸We can also describe the impact of productivity, at least if θ is large, so uniqueness is guaranteed (details are in an Online Appendix). Suppose $F^j(k, h) = B \varepsilon_j f(k) + h$, and note that output y moves in the same direction as k , since in steady state $x + \delta k = y$, where $u'(x) = A$. Hence, we identify higher k with higher y and better economic times. Then, we can show k goes up with B , ε_H , ε_L and γ_H . At least under Condition 2, q also goes up B and ε_H , although the impact of ε_L and γ_H on q are ambiguous. Of course, these effects, like the policy effects discussed in the text, are derived for a quasi-linear technology, so w is fixed; more general technologies entail more general equilibrium complications, but the forces we identify are still clearly relevant.

Proposition 6 Consider $F(k, h) = \varepsilon f(k) + h$ with bargaining and assume equilibrium exists. If $\theta = \theta^*$ then equilibrium is efficient if $\iota = \tau = 0$. For $\theta \neq \theta^*$ it is efficient if $\iota = 0$ and $\tau = \tau^*$, where $\tau^* < 0$ if $\theta > \theta^*$ and $\tau^* > 0$ if $\theta < \theta^*$.

We end this section with a result on optimal monetary policy under $\tau = 0$.

Proposition 7 Suppose $F(k, h, \varepsilon) = \varepsilon f(k) + h$. If $\tau = 0$, there exists two cutoffs $0 < \theta_L \leq \theta^* \leq \theta_H < 1$ such that the optimal monetary policy is $\iota^* > 0$ if $\theta > \theta_H$ or $\theta < \theta_L$. In either cases, the first best is not achieved. If in addition, $f(k) = k^\eta$ where $\eta \in (0, 1)$, then

$$\theta_L = \theta^* = \frac{\varepsilon_H - \varpi^{1-\eta}}{\varepsilon_H - \varepsilon_L},$$

$$\theta_H = \frac{\varepsilon_H - \varepsilon_L + \varpi^{1-\eta} - \sqrt{[\varepsilon_H - \varepsilon_L + \varpi^{1-\eta}]^2 - 2 \frac{\varpi^{1-\eta} \varepsilon_H^{\frac{1}{1-\eta}} (\varepsilon_H - \varepsilon_L)}{\varpi}}}{2(\varepsilon_H - \varepsilon_L)},$$

where $\varpi = \varepsilon_H^{\frac{1}{1-\eta}}/2 + \varepsilon_L^{\frac{1}{1-\eta}}/2$. Moreover, $\iota = 0$ is a local maximizer of the welfare function if $\theta \in (\theta_L, \theta_H)$. the total welfare is decreasing at $\iota = 0$ if $\theta \in (\theta_L, \theta_H)$.

Proposition 7 shows that the Friedman rule may not be optimal if $\tau = 0$. This happens if θ is sufficiently large or sufficiently small. In the former case, firms under-invest because of the purchase opportunity in the DM. Higher ι decreases the value of this purchase option and induces more investment, which leads to higher welfare. In the latter, firms over-invest because of the resale opportunity. Higher ι reduces the resale value of capital and reduces investment. In either cases, the first best is not achieved at the optimal monetary policy because the MPK does not equalize across buyers and sellers in the DM, which is a required for efficiency. If, in addition, $f(k) = k^\eta$, we can obtain closed-form solution for θ_L and θ_H . Interestingly, θ^* equals θ_L , i.e. the Friedman rule is not optimal at any $\theta < \theta^*$. Moreover, we can show that the Friedman rule is locally optimal

if $\theta \in (\theta_L, \theta_H)$. We are not able to show that the Friedman rule maximizes welfare globally in general. But in the numerical examples considered, the welfare functions are all concave and the Friedman rule is the optimal monetary policy.

6 Competitive Search

We now consider some different ways of organizing the secondary capital market, using directed search and price posting instead of random search and bargaining. One reason is that both specifications have proved useful in various applications in, e.g., labor, housing, finance and monetary economics – they are simply different ways to formalize the notion of markets with frictions – and we want to know what each implies about capital investment and reallocation. For this exploration, we again use the convenient parameterizations with quasi-linear technology and a two-point distribution of shocks, $\varepsilon \in \{\varepsilon_L, \varepsilon_H\}$.

As discussed in the survey by Wright et al. (2017), the solution concept of posting plus directed search is called competitive search equilibrium, because those who post the terms of trade compete to attract customers, which by assumption means commitment (there are no renegotiations in meetings). One way to describe this in a two-sided market is to say that sellers post the terms of trade and buyers direct their search to the sellers they find most attractive, which does not necessarily mean those with the lowest price, since one has to take into account the probability of trade (meeting an appropriate counterparty) and that depends on the buyer/seller ratio, or market tightness. Another is to say that buyers post and sellers search. Yet another is to say that third parties called market makers post terms to attract both buyers and sellers to submarkets of their design, the idea being that they can earn profits by charging participation fees, but free entry drives these fees to 0. All three approaches yield the same

outcome in most standard environments, although there are exceptions (e.g., Faig and Huangfu 2007; Delacroix and Shi 2017).

We consider two ways of setting up the model. In the first, market makers post the terms of trade after the shocks are realized, at which point we have a standard two-sided market, so it is equivalent to let sellers post to attract buyers, or vice versa. It turns out, however, that this does not do all that well at getting agents to bring to market the efficient (\hat{z}, \hat{k}) , because the costs of investing in money and capital are sunk when the terms of trade are posted. So we also consider a second setup where market makers post the terms of trade before the shocks are realized, which does better in terms of (\hat{z}, \hat{k}) , but is slightly nonstandard because agents decide where to search before knowing whether they will be buyers or sellers. While that is not a big problem, it raises some issues, as discussed below.

To begin, consider the scenario where market makers design submarkets in the DM by posting (p, q, n) in the DM, after the shocks are realized, where n is tightness in the particular submarket. In equilibrium, there will turn out that all active submarkets have the same $n = \gamma_H/\gamma_L$, given by the ratio of buyers to sellers in the population, but in principle a market maker can choose a different n . Actually, it is not important to post n , since agents can figure it out by observing p and q , but as is standard, for convenience, we proceed as if it is posted. Then, after comparing (p, q, n) in all the submarkets, buyers and sellers decide which one to visit.

We consider the alternative, where agents decide where to go before seeing the ε shocks. If they visit a submarket posting (k, q, z, n) they must bring capital k and real balances z . Then, after they are partitioned into sets of buyers and sellers by the shocks, they engage in random meetings, where the meeting probabilities are $\alpha_L = \alpha\gamma_H$ and $\alpha_H = \alpha\gamma_L$, as above. If a buyer with ε_H meets a seller with

ε_L , they are committed to trading q units of capital for a payment of z units of real balances as well as $(1 - \delta)q$ in secured debt.¹⁹

For simplicity, let $A = B = 1$ and $\tau = 0$. The market maker posts (p, q, n) in the DM given (z, k_b, k_s) where k_b and k_s are the capital stock of buyers and sellers, respectively. Buyers and sellers choose which submarket to enter. Then the MM's problem is

$$\begin{aligned} v_b &= \max_{p,q,n} \frac{\alpha(n)}{n} \{ \varepsilon_H [f(k_b + q) - f(k_b)] - p \} + (1 - \delta)k_b + W_t(z, 0, \varepsilon_H), \\ \text{st} \quad &: \quad v_s = \alpha(n) \{ p - \varepsilon_L [f(k_s) - f(k_s - q)] \} + (1 - \delta)k_s + W_t(z, 0, \varepsilon_L) \\ & z \geq p \end{aligned}$$

Let

$$\begin{aligned} \bar{v}_b &= v_b - (1 - \delta)k_b - W_t(z, 0, \varepsilon_H), \\ \bar{v}_s &= v_s - (1 - \delta)k_s - W_t(z, 0, \varepsilon_L). \end{aligned}$$

Since $W_t(z, 0, \varepsilon_H)$, $W_t(z, 0, \varepsilon_L)$, $(1 - \delta)k_b$, and $(1 - \delta)k_s$ are constants, then one can rewrite the problem as

$$\begin{aligned} \bar{v}_b &= \max_{p,q,n} \frac{\alpha(n)}{n} \{ \varepsilon_H [f(k_b + q) - f(k_b) - p] \}, \\ \text{st} \quad &: \quad \bar{v}_s = \alpha(n) \{ p - \varepsilon_L [f(k_s) - f(k_s - q)] \} \\ & z \geq p. \end{aligned}$$

If $z \geq p$ is not binding, one can substitute in p from the constraint and obtain

$$\bar{v}_b = \max_{p,q,n} \frac{\alpha(n)}{n} \{ \varepsilon_H [f(k_b + q) - f(k_b) - \varepsilon_L [f(k_s) - f(k_s - q)]] \} - \frac{\bar{v}_s}{n}.$$

¹⁹An alternative formulation has market makers posting (q, p, n) , while agents choose k and m , respectively, subject to the constraints $q \leq k$ and $p \leq m$; the results are the same.

FOC yields

$$\begin{aligned} q & : \quad \varepsilon_H f'(k_b + q) - \varepsilon_L f'(k_s - q) = 0 \\ n & : \quad \alpha(n) [1 - \varepsilon(n)] \{ \varepsilon_H [f(k_b + q) - f(k_b)] - \varepsilon_L [f(k_s) - f(k_s - q)] \} = \bar{v}_s \end{aligned}$$

And $n = \gamma_H/\gamma_L$ pins down \bar{v}_s and hence p .

If $z \geq p$ is binding, then using the langragian to obtain

$$\max_{q,n} \frac{\alpha(n)}{n} \{ \varepsilon_H [f(k_b + q) - f(k_b)] - z \} + \lambda \{ \alpha(n) \{ z - \varepsilon_L [f(k_s) - f(k_s - q)] \} - \bar{v}_s \}.$$

FOC yields

$$\begin{aligned} q & : \quad \frac{\alpha(n)}{n} \varepsilon_H f'(k_b + q) - \lambda \alpha(n) \varepsilon_L f'(k_s - q) = 0 \\ n & : \quad \frac{\alpha'(n) n - \alpha(n)}{n^2} \{ \varepsilon_H [f(k_b + q) - f(k_b)] - z \} + \lambda \alpha'(n) \{ z - \varepsilon_L [f(k_s) - f(k_s - q)] \} = 0 \end{aligned}$$

Solve for λ from the first equation.

$$\frac{1}{n} \frac{\varepsilon_H f'(k_b + q)}{\varepsilon_L f'(k_s - q)} = \lambda$$

and substitute it into the second equation to obtain

$$- [1 - \varepsilon(n)] \{ \varepsilon_H [f(k_b + q) - f(k_b)] - z \} \varepsilon_L f'(k_s - q) + \varepsilon(n) \varepsilon_H f'(k_b + q) \{ z - \varepsilon_L [f(k_s) - f(k_s - q)] \} = 0$$

which yields

$$\begin{aligned} z & = \quad p = Z(q, k_b, k_s) \\ & = \quad \frac{[1 - \varepsilon(n)] \varepsilon_L f'(k_s - q) \varepsilon_H [f(k_b + q) - f(k_b)] + \varepsilon(n) \varepsilon_H f'(k_b + q) \varepsilon_L [f(k_s) - f(k_s - q)]}{\varepsilon(n) \varepsilon_H f'(k_b + q) + [1 - \varepsilon(n)] \varepsilon_L f'(k_s - q)} \end{aligned}$$

where $n = \gamma_H/\gamma_L$ and $\varepsilon(n) = n\alpha'(n)/\alpha(n)$. Therefore, we have

$$\begin{aligned} \frac{\partial q}{\partial z} & = \frac{1}{Z_1(q, k_b, k_s)}, \\ \frac{\partial q}{\partial k_b} & = - \frac{Z_2(q, k_b, k_s)}{Z_1(q, k_b, k_s)}, \\ \frac{\partial q}{\partial k_s} & = - \frac{Z_3(q, k_b, k_s)}{Z_1(q, k_b, k_s)} \end{aligned}$$

Then we solve for the CM problem.

$$W_t(a, k, \varepsilon) = \max_{x, h, \hat{z}, \hat{k}} \{u(x) - h + \beta \mathbb{E}_{\hat{\varepsilon}} V_{t+1}(\hat{z}, \hat{k}, \hat{\varepsilon})\}$$

$$\text{st } x + \phi_t \hat{z} / \phi_{t+1} + \hat{k} = h + a + (1 - \delta)k + B(1 - \tau) \varepsilon f(k) - T.$$

Then FOC gives

$$\phi_t / \phi_{t+1} = \beta \frac{\partial}{\partial \hat{z}} \mathbb{E}_{\hat{\varepsilon}} V_{t+1}(\hat{z}, \hat{k}, \hat{\varepsilon})$$

$$1 = \beta \frac{\partial}{\partial \hat{k}} \mathbb{E}_{\hat{\varepsilon}} V_{t+1}(\hat{z}, \hat{k}, \hat{\varepsilon}).$$

In addition,

$$V_{t+1}(\hat{z}, \hat{k}, \varepsilon_H) = \frac{\alpha(n)}{n} \left\{ \varepsilon_H \left[f(\hat{k} + q) - f(\hat{k}) \right] - \hat{z} \right\} + (1 - \delta) \hat{k} + W_t(z, 0, \varepsilon_H)$$

$$V_{t+1}(\hat{z}, \hat{k}, \varepsilon_L) = \alpha(n) \varepsilon_L \left\{ \hat{z} - \varepsilon_L \left[f(\hat{k}) - f(\hat{k} - q) \right] \right\} + (1 - \delta) \hat{k} + W_t(z, 0, \varepsilon_L)$$

Therefore, in the steady state, we have

$$\iota = \gamma_H \frac{\alpha(n)}{n} \left[\frac{\varepsilon_H f'(k + q)}{Z_1(q, k, k)} - 1 \right],$$

$$r + \delta = \frac{\alpha(n)}{n} \gamma_H \varepsilon_H f'(k + q) \left[1 - \frac{Z_2(q, k, k)}{Z_1(q, k, k)} \right] + \alpha(n) \gamma_L \varepsilon_L f'(k - q) \left[1 + \frac{Z_3(q, k, k)}{Z_1(q, k, k)} \right]$$

$$+ \left[1 - \frac{\alpha(n)}{n} \right] \gamma_H \varepsilon_H f'(k) + [1 - \alpha(n)] \gamma_L \varepsilon_L f'(k).$$

We know from the Nash solution that $\iota = 0$ does that imply $q = q^*(k)$

The problem for a market maker is

$$v = \max_{k, q, z} \left\{ - (r + \delta) k - \iota z + \frac{\alpha(n)}{n} \gamma_H S^H(k, z, q) \right. \quad (37)$$

$$\left. + \alpha(n) \gamma_L S^L(k, z, q) + (\gamma_H \varepsilon_H + \gamma_L \varepsilon_L) f(k) \right\}$$

$$\text{st } 0 \leq S^H(k, z, q), 0 \leq S^L(k, z, q),$$

where

$$\begin{aligned} S^H(k, z, q) &= \varepsilon_H(1 - \tau)B[f(k + q) - f(k)] - z, \\ S^L(k, z, q) &= z - \varepsilon_L(1 - \tau)B[f(k) - f(k - q)] \end{aligned}$$

are the surpluses S^H and S^L of buyers and sellers in a match, and they we constrain them to be nonnegative.²⁰

It is easy to show that the objective function is decreasing in z . Therefore, it is optimal to choose z such that the seller's IC constraint is binding, i.e., $S^L(k, z, q) = 0$. Then it is obvious that $S^H(k, z, q) > 0$. As usual, we can use $S^L(k, z, q) = 0$ to eliminate z from the objective function. Then one can follow the standard approach to show that the equilibrium conditions for (k, q) are

$$\iota = \frac{\alpha(n)}{n} \gamma_H \frac{\varepsilon_H f'(k + q) - \varepsilon_L f'(k - q)}{\varepsilon_L f'(k - q)}, \quad (38)$$

$$\begin{aligned} \frac{r + \delta}{B(1 - \tau)} &= (\gamma_H \varepsilon_H + \gamma_L \varepsilon_L) f'(k) + \frac{\alpha(n)}{n} \gamma_H \varepsilon_H [f'(k + q) - f'(k)] \\ &\quad + [\alpha(n) \gamma_L + \iota] \varepsilon_L [f'(k - q) - f'(k)], \end{aligned} \quad (39)$$

$$n = \gamma_H / \gamma_L.$$

²⁰This is a subtle problem. Faig and Huangfu (2007) were the first to point out that a market maker can potentially run the following clever scheme in a model with ex ante heterogeneity across households trading goods: all buyers that visit the submarket give the market maker their cash; then if a buyer meets a seller the latter gives the former the good for free; and finally, when agents exit the submarket the market maker gives the cash to the sellers. This allows buyers to share the cost of carrying cash when $\iota > 0$ – i.e., to provide liquidity insurance, something like a bank in Diamond and Dybvig (1983), or better yet Berentsen et al. (2007), which is a monetary model. Rocheteau and Wright (2005) rule out this scheme by saying the market maker cannot tell buyers from seller at the exit stage, but that is somewhat ad hoc. In any event, our problem is even more involved due to ex post heterogeneity. A market maker can in principle tell all participants to show up with no cash, then if a low- meets a high-productivity firm in the matching process, the former gives the latter q for free. This perfect insurance arrangement, which avoids cash entirely, is nice, but we want to rule it out. To do so we proceed in the spirit of mechanism design a la Hu et al. (2009). This allows the market maker, like a mechanism designer, to suggest the terms of trade, and agents who meet implement them as long as they are in the bilateral core. This means agents cannot get negative surplus, else they would unilaterally deviate, and the pair cannot have a profitable bilateral deviation. In our context a firm realizing ε_L would like to commit to deliver q to one realizing ε_H if they meet, but cannot. Hence, he will only turn over q if he gets something in quid pro quo, and this is where the money comes back in.

In addition, the equilibrium z can be calculated from seller's IC condition,

$$\frac{z}{(1-\tau)B} = \varepsilon_L [f(k) - f(k-q)].$$

Then we have the following results:

Proposition 8 *Given $F(k, h, \varepsilon) = \varepsilon f(k) + h$ and $\varepsilon \in \{\varepsilon_H, \varepsilon_L\}$, with competitive search, a monetary steady state exists if $\iota < \hat{\iota} = \alpha_H \gamma_H (\varepsilon_H - \varepsilon_L) / \varepsilon_L$. It is unique if ι is sufficiently small.*

As in Proposition 5, a monetary equilibrium exists only if ι is below a threshold. But now the threshold is bigger, unless $\theta = 1$, because competitive search is a more efficient trading arrangement. Figure 7 shows the CR and IS curves in this case as well as the policy effects. The IS curve is nonmonotone: it decreases with q if q is small and then increases with q when it is big, similar to the bargaining model with θ not too big or small. But uniqueness always obtains if ι is not too big.

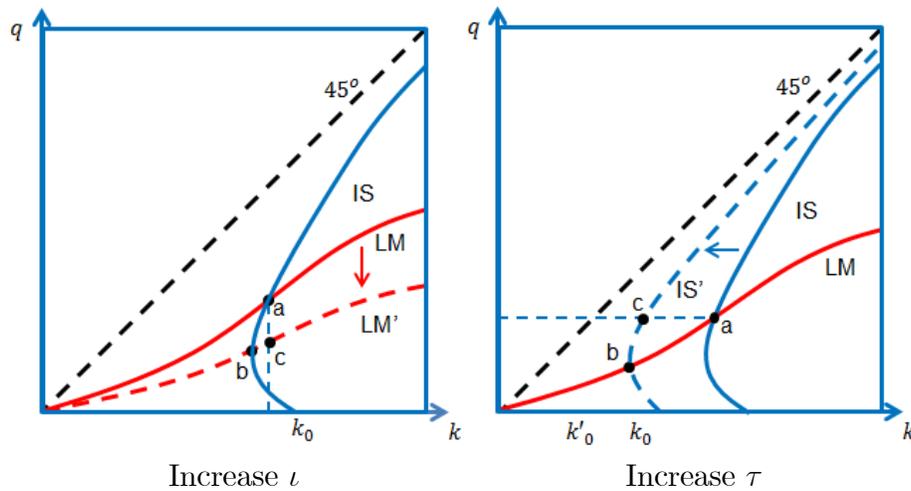


Figure 7: Competitive Search Equilibrium

In terms of policy, at least for ι not too big we can show this (see the Online Appendix for details): Increases in ι reduce k at least under Condition 1, and always reduce q . Also, increases in τ always reduce k , and reduce q at least under Condition 2.²¹ The next result concerns efficiency, and can be proved simply by comparing (38)-(39) to the planner's solution.

Proposition 9 *Given $F^j(k, h) = f^j(k) + h$, with competitive search, and $\tau = 0$, equilibrium is efficient iff $\iota = 0$.*

Two aspects of Proposition 9 are noteworthy. First, (38) looks like what one gets with Nash bargaining when $\theta = 1$, which is what it takes to get efficient q at $\iota = 0$. Finally, (39) looks like that the buyer gets what he can get with Nash bargaining when $\theta = 1$ and that the seller gets what he can get with $\theta = 0$, which is what it takes to get efficient k . Thus competitive search avoids the holdup problems inherent in bargaining and makes it easier to achieve efficiency. We do not take a stand on which solution concept is “better.” The goal instead is sort out logically the inefficiencies that obtain under different market structures and the implications for policy.

7 Numerical Analysis

To further study the property of the equilibrium with bargaining, this section shows some numerical examples. We start by showing that the model is consistent with stylized facts of capital reallocation in the literature: reallocation is procyclical; mismatch is countercyclical; price of used capital is procyclical; and spending on use capital as a fraction of total investment spending is procyclical.

²¹In terms of productivity, we can show increases in B raise k and raise q at least under Condition 2. Also, increases in ε_H raise both, while increases in ε_L raise k and reduce q at least under Condition 1. Also, increases in γ_H raise both k and q at least under Condition 1. Again, we need ι not too big for some of these results.

In addition, Kehrig (2015) shows that the dispersion of productivity is counter-cyclical with the lower end more effected by the business cycles.²² We are going to take this as exogenous and use the model to generate the four stylized facts. Intuitively, increasing both ε_H and ε_L leads to higher q , so there is more reallocation in good times. If ε_L goes up by more than ε_H , then mismatch is lower in good times.

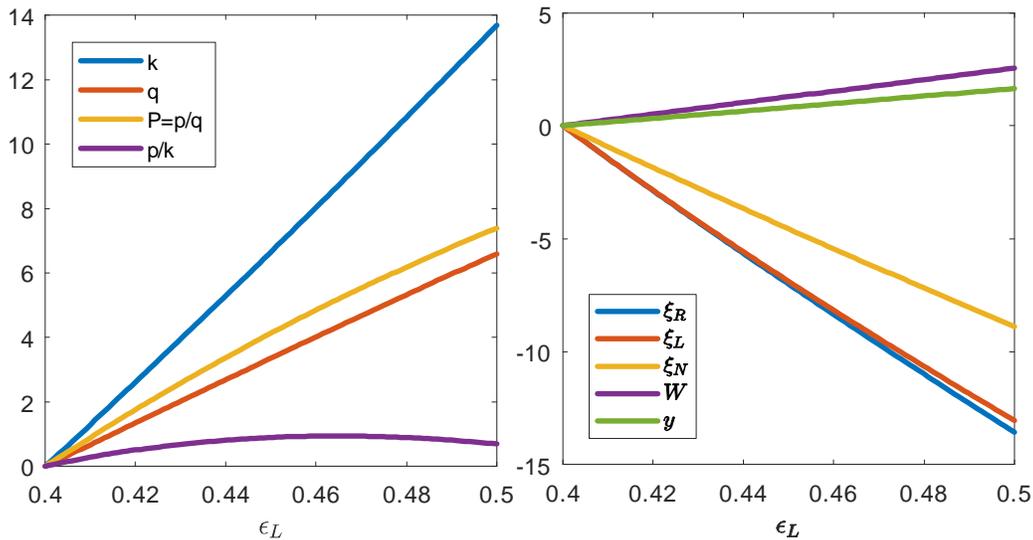


Figure 4: Volume, Price and MPK Dispersion.

Figure 4 shows such an example.²³ It indicates how ε_L affects the outcome in terms of k , q , $P = p/q$ and p/k , as well as welfare W measured in the standard way as a consumption equivalent, plus three notions of dispersion defined by the cross-sectional standard deviation of marginal products, MPK , of $\log MPK$, and of MPK normalized by the average, denoted by ξ , $\hat{\xi}$, and $\tilde{\xi}$, respectively. As shown, good times are associated with less mismatch by all three measures. Still,

²²Kehrig (2015), e.g., establishes these results: “First, crosssectional productivity dispersion is countercyclical; the distribution of total factor productivity levels across establishments is about 12% more spread-out in a recession than in a boom. Second, the bottom quantiles of the productivity distribution are more cyclical than the top quantiles. In other words, the countercyclical of productivity dispersion is mostly due to a higher share of relatively unproductive establishments during downturns.”

²³The example uses $u(x) = 4x^{0.5}$, $f(k) = k^{0.3}$, $\alpha = 0.5$, $\gamma_H = \gamma_L = 0.5$, $\beta = 0.98$, $\delta = 0.05$, $\tau = 0$, $\theta = 1$, $A = B = C = 1$, $\iota = 0.01$ and $\varepsilon_H = 2.3$, while ε_L varies along the horizontal axis.

good times have more reallocation measured by DM spending p , the quantity traded q , and spending as a fraction of the stock p/k . Also, good times have a higher unit price of capital, $P = p/q$.

So the model has no problem matching the key stylized facts. Moreover, many economists (e.g., Eisfeldt and Rampini 2006; Hsieh and Klenow 2009) interpret misallocation in this way – marginal products are not equalized across firms. It is therefore tempting to think these statistics are positively associated with some notion of frictions and negatively associated with welfare. But this need not be true, as we now discuss.

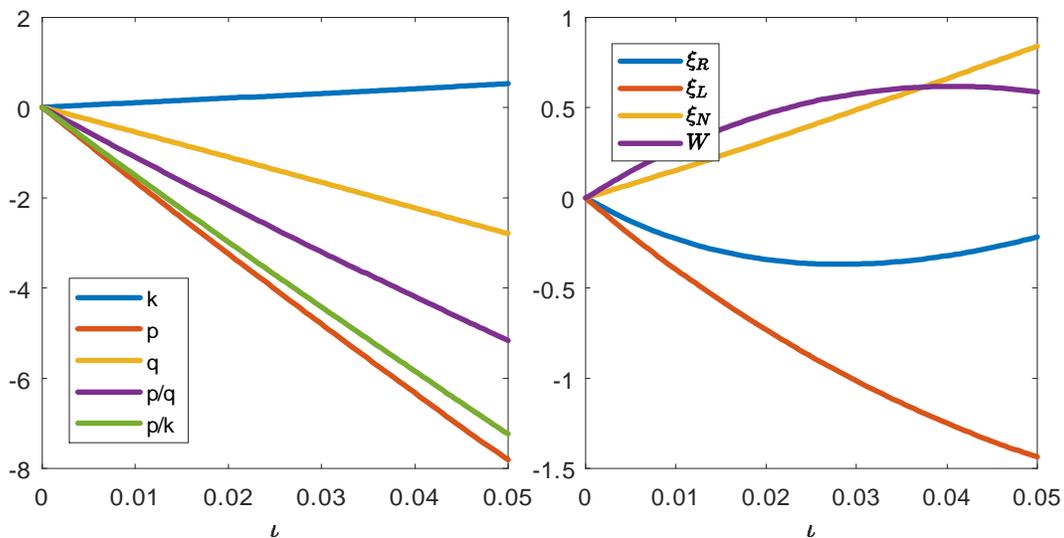


Figure 5: Misallocation, k , q , P , z/k and W vs ι with $\theta = 1$

The nominal rate is a financial fiction: when ι is higher liquidity constraints are tighter. Figures 5 and 6 show, for different θ , how ι affects the various measures of misallocation, plus k , q , p/q , p/k and W .²⁴ Clearly, ξ need not increase in ι if $\theta = 1$. This is because higher ι has two effects, as shown in the left

²⁴The example uses $u(x) = 4x^{0.5}$, $f(k) = k^{0.3}$, $\alpha = 0.5$, $\gamma_H = \gamma_L = 0.5$, $\beta = 0.98$, $\delta = 0.05$, $\tau = 0$, $A = B = C = 1$, $\varepsilon_L = 0.4$ and $\varepsilon_H = 3$, while ι varies along the horizontal axis and θ are indicated in the captions.

panel of Figure 5: it reduces q but increases k , which homogenizes MPK (e.g., if all firms have very large k , their marginal products are all close to 0, regardless of the ε 's). If the latter effect dominates the former, which it does when $\theta = 1$, then ξ decreases with ι . However, if θ is small then, as shown in Figure 6, both k and q are decreasing and ξ is monotonically increasing in ι . The other commonly used measure $\hat{\xi}$ performs even worse in terms of capturing frictions, as it is decreasing in ι in both examples. The measure $\tilde{\xi}$ performs better – in these examples, it is increasing in ι – because it corrects for the homogenization effect of higher k .

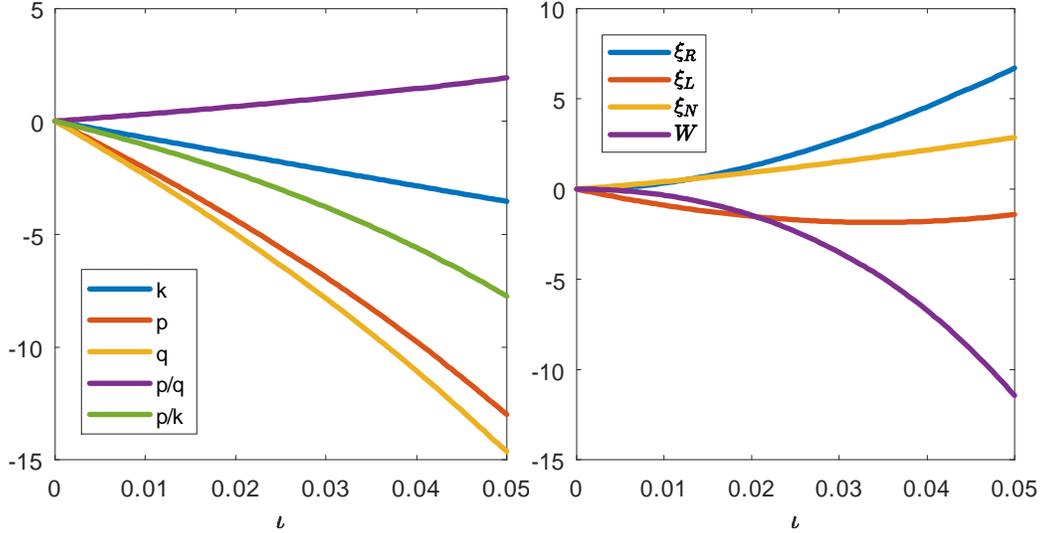


Figure 6: Misallocation, k , q , y and welfare vs ι with $\theta = 0.423$.

When it comes to welfare, none of these misallocation measures do well. In Figure 5, W increases for ι between 0.03 and 0.04 while ξ and $\tilde{\xi}$ are increasing. In Figure 6, W is globally decreasing in ι while $\hat{\xi}$ decreases between 0 and 0.03. We also documented similar results for the search frictions but do not include them in the interest of space. The lesson is that when the economy has fundamental frictions, less dispersion in MPK or $\log MPK$ does not necessarily correspond to lower efficiency or higher frictions. While the standard deviation of MPK

divided by the average MPK may be a better measure of frictions, it is not necessarily monotone in welfare.²⁵

Note that $\theta = 1$ implies W is increasing in ι at $\iota = 0$, and so the optimal policy is not the Friedman rule. This is because there is under accumulation of capital when θ is high, and that potentially makes $\iota > 0$ beneficial, because it encourages CM accumulation. Alternatively, as in the ex ante model we could subsidize investment by lowering τ , which happens to be 0 in this example. Given $\tau = 0$, the optimal monetary policy is achieved at $\iota = 4.1\%$, corresponding to an inflation rate of $\pi = 2\%$. While by no means a serious calibration, the example uses very reasonable parameters, and delivers realistic nominal interest and inflation rate targets. If instead of $\theta = 1$, if we set $\theta = 0.423$, which is the particular value defined in (35), the optimal policy is $\iota = \tau = 0$.

8 Conclusion

This paper has explored the determination of capital investment and reallocation in dynamic general equilibrium. The theory included frictional secondary markets, with monetary exchange, and different microstructures, including ex random search and bargaining as well as directed search and posting. For each specification we provided results on existence, uniqueness, efficiency and policy. The framework is tractable: it can be reduced to two equations for capital and money – i.e., for investment and reallocation. Depending on parameters, decreasing the nominal interest rate stimulates real investment and output, consistent with some mainstream macroeconomic thinking, even if our approach to microfoundations

²⁵These findings are related to Asker et.al. (2014) who argues that dispersion of MPK can be dynamically efficient and purely an outcome of time-to-build assumption of capital and productivity uncertainties. Their result requires that there is no secondary market after productivity reveals. We argue that with a frictional secondary market, the higher the dispersion of MPK does not indicate higher frictions nor lower welfare.

is rather different.

In some versions of the model inflation above the Friedman rule is optimal because, while it hinders the secondary market, it encourages investment in the primary market, and on net this could be desirable. We also argued that common measures of mismatch related to productivity dispersion do not necessarily capture market frictions – e.g., higher inflation can reduce the dispersion of marginal products. Further, we showed how to account for the observation that reallocation is procyclical while mismatch is countercyclical. All these results help us better understand issues related to investment and reallocation, and to the effects of monetary and fiscal policy.

In terms of future research, one could pursue the quantitative analysis more thoroughly. For this one could relax assumptions such as having i.i.d. shocks or having only two realizations of the shocks. One can also add aggregate shocks. It might be interesting to examine endogenous growth in this framework, perhaps allowing liquid assets other than currency to facilitate trade, and allowing financial intermediation. Also, it might be interesting to combine models with frictional capital and frictional labor markets. All of this is left for future work.

Appendix

Proof of Proposition 5: First, (33) defines q as a function of k , say $q = Q(k)$, as long as ι is not too big, where

$$\begin{aligned} Q(k) &= \frac{f'(k+q)f''(k-q) - f'(k-q)f''(k+q)}{f'(k+q)f''(k-q) + f'(k-q)f''(k+q)} \\ &\simeq \frac{\partial}{\partial q} [f'(k+q)f'(k-q)]. \end{aligned}$$

Although $f'(k+q)f'(k-q)$ is increasing in common examples like $f(k) = k^\eta$, it cannot be signed in general. Notice $Q(0) = 0$, $Q(k) > 0$ if $k > 0$, and $Q'(k) < 1$.

Similarly, (34) defines k as a function of q : $k = K(q)$. Let k_0 satisfy

$$\frac{r + \delta}{1 - \tau} = (\gamma_H \varepsilon_H + \gamma_L \varepsilon_L) f'(k_0).$$

Then $K(0) = k_0$. In addition, let \bar{k} satisfy

$$\begin{aligned} \frac{r + \delta}{1 - \tau} &= \gamma_H \alpha_H \theta \varepsilon_H f'(2\bar{k}) + \gamma_H \varepsilon_H (1 - \alpha_H) f'(\bar{k}) \\ &\quad + \frac{\gamma_L \alpha_L (1 - \theta)}{\theta} \varepsilon_H f'(2\bar{k}) + \gamma_L \varepsilon_L f'(\bar{k}). \end{aligned}$$

Then $\bar{k} = K(\bar{k})$. Any q solves $Q \circ K(q) = q$ is an equilibrium. Notice, $Q \circ K(0) = Q(k_0) > 0$ and $Q \circ K(\bar{k}) = Q(\bar{k}) < \bar{k}$. Then by continuity, there exists at least one equilibrium..

If $\theta = 1$, one can easily check the FOC's are sufficient. Therefore, (k, q) is a steady state equilibrium iff it satisfies

$$\iota = \alpha_H \gamma_H \left[\frac{\varepsilon_H f'(k+q)}{\varepsilon_L f'(k-q)} - 1 \right] \quad (40)$$

$$r + \delta = \gamma_H \alpha_H \varepsilon_H f'(k+q) + (1 - \alpha_H) \gamma_H \varepsilon_H f'(k) + \gamma_L \varepsilon_L f'(k). \quad (41)$$

Uniqueness follows if $K'(q) Q' \circ K(q) < 0$ whenever $Q \circ K(q) = 0$. To check this,

notice

$$\begin{aligned}
K_2'(q) Q_1' \circ K_2(q) &\simeq -f'(k+q) f''(k-q) \gamma_H \alpha_H \varepsilon_H f''(k+q) \\
&\quad - \gamma_H \alpha_H \varepsilon_H f''(k+q) f'(k-q) f''(k+q) \\
&\quad - [f'(k+q) f''(k-q) + f'(k-q) f''(k+q)] \\
&\quad \times [(1-\alpha_H) \gamma_H \varepsilon_H f''(k) + \gamma_L \varepsilon_L f'(k)] < 0.
\end{aligned}$$

This proves uniqueness.

$$\iota = \alpha_H \gamma_H \theta [\varepsilon_H f'(k+q) - \varepsilon_L f'(k-q)] / D \quad (42)$$

$$\begin{aligned}
\frac{r+\delta}{1-\tau} &= \gamma_H \varepsilon_H [\alpha \gamma_L f'(k+q) \Omega(k,q) + (1-\alpha \gamma_L) f'(k)] \quad (43) \\
&\quad + \gamma_L \varepsilon_L [\alpha \gamma_H f'(k-q) \Gamma(k,q) + (1-\alpha \gamma_H) f'(k)],
\end{aligned}$$

$$\Omega(k,q) = [(1-\theta) \varepsilon_H f'(k) + \theta \varepsilon_L f'(k-q)] / D$$

$$\Gamma(k,q) = [(1-\theta) \varepsilon_H f'(k+q) + \theta \varepsilon_L f'(k)] / D$$

If $\theta > 0$ and $\iota = 0$, (33) and (34) reduces to

$$\begin{aligned}
0 &= \varepsilon_H f'(k+q) - \varepsilon_L f'(k-q) \\
\frac{r+\delta}{1-\tau} &= \gamma_H \{ \alpha \gamma_L [(1-\theta) \varepsilon_H f'(k) + \theta \varepsilon_L f'(k-q)] + (1-\alpha \gamma_L) \varepsilon_H f'(k) \} \\
&\quad + \gamma_L \{ \alpha \gamma_H [(1-\theta) \varepsilon_H f'(k+q) + \theta \varepsilon_L f'(k)] + (1-\alpha \gamma_H) \varepsilon_L f'(k) \}.
\end{aligned}$$

Partially differentiate the right hand side of the above equations with respect to q and k to obtain

$$\begin{aligned}
\hat{\mathbf{J}}_{11} &= \varepsilon_H f'(k+q) + \varepsilon_L f''(k-q) \\
\hat{\mathbf{J}}_{12} &= \varepsilon_H f''(k+q) - \varepsilon_L f''(k-q) \\
\hat{\mathbf{J}}_{21} &= \alpha \gamma_L \gamma_H \{ (1-\theta) [\varepsilon_H f''(k) + f''(k+q)] - \theta \varepsilon_L [f''(k-q) - f''(k)] \} \\
\hat{\mathbf{J}}_{22} &= \alpha \gamma_L \gamma_H \{ (1-\theta) [\varepsilon_H f''(k) + f''(k+q)] + \theta \varepsilon_L [f''(k-q) + f''(k)] \} \\
&\quad + \varepsilon_H \gamma_H (1-\alpha \gamma_L) f''(k) + \varepsilon_L \gamma_L (1-\alpha \gamma_H) f''(k).
\end{aligned}$$

Obviously, $\hat{\mathbf{J}}_{11} < 0$ and $\hat{\mathbf{J}}_{22} < 0$. Therefore, one can show that at the equilibrium k and q ,

$$K'_2(q) Q'_1 \circ K_2(q) - 1 = \frac{\hat{\mathbf{J}}_{12} \hat{\mathbf{J}}_{21}}{\hat{\mathbf{J}}_{11} \hat{\mathbf{J}}_{22}} - 1 \simeq \hat{\mathbf{J}}_{12} \hat{\mathbf{J}}_{21} - \hat{\mathbf{J}}_{11} \hat{\mathbf{J}}_{22} < 0$$

which implies uniqueness follows for $\iota = 0$. Then by continuity, uniqueness holds for ι not too big. ■

Proof of Proposition 8 Equation (38) defines q as a continuous function of k , i.e., $q = Q(k)$ where

$$Q'(k) = - \frac{f''(k+q) f'(k-q) - f'(k+q) f''(k-q)}{f''(k+q) f'(k-q) + f'(k+q) f''(k-q)}$$

can be positive or negative depends on the sign of $f''(k+q) f'(k-q) - f'(k+q) f''(k-q)$.

In addition, if $k \rightarrow 0$, $Q(k) \rightarrow 0$ and if $k \rightarrow \infty$, $Q(k) \rightarrow \infty$ and $k - Q(k) \rightarrow \infty$.

Equation (39) defines k as a continuous function of q : $k = K(q)$. This function may not be a monotone function because

$$K'(q) \simeq - \left\{ \frac{\alpha(n)}{n} \gamma_H \varepsilon_H f''(k+q) - \left[\frac{\alpha(n)}{n} \gamma_H + \iota \right] \varepsilon_L f''(k-q) \right\} \leq 0.$$

Notice that $K(0) = k_0$ where k_0 solves

$$\frac{r + \delta}{(1 - \tau) B} = (\gamma_H \varepsilon_H + \gamma_L \varepsilon_L) f'(k_0).$$

If $q \rightarrow \infty$, $K(q) \rightarrow \infty$ and $K_2(q) - q \rightarrow c < \infty$ where c solves

$$\frac{r + \delta}{(1 - \tau) B} = [\alpha(n) \gamma_L + \iota] \varepsilon_L f'(c).$$

Any k that satisfies $K \circ Q(k) - k = 0$ is an equilibrium. Notice $K \circ Q(0) = k_0 > 0$.

In addition, if $k \rightarrow \infty$

$$K \circ Q(k) - k = K \circ Q(k) - Q(k) + Q(k) - k \rightarrow c - \infty.$$

This means that $K \circ Q(k) - k < 0$ for k sufficiently large. By the intermediate

value theorem, an equilibrium exists. At the equilibrium k and $q = Q(k)$,

$$\begin{aligned}
& \frac{\partial}{\partial k} [K \circ Q(k) - k] \\
= & K' \circ Q(k) Q'(k) - 1 \\
\simeq & \iota \varepsilon_L f''(k) \frac{\partial}{\partial k} f'(k+q) f'(k-q) - 2\iota \varepsilon_L f''(k-q) f''(k+q) f'(k-q) \\
& - \left\{ \gamma_H \left[1 - \frac{\alpha(n)}{n} \right] \varepsilon_H + \gamma_L [1 - \alpha(n)] \varepsilon_L \right\} f''(k) \frac{\partial}{\partial k} f'(k+q) f'(k-q) \\
& - 2 \frac{\alpha(n)}{n} \gamma_H \varepsilon_H f''(k+q) \frac{\partial}{\partial k} f'(k+q) f'(k-q).
\end{aligned}$$

Notice all but the first term are negative. Therefore, if ι sufficiently small, $K' \circ Q(k) Q'(k) - 1 < 0$. In this case, $K \circ Q(k) - k = 0$ for at most one k and uniqueness follows. ■

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